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MATHEMATICAL TREATISES,

CONTAINING

I. THE THEORY OF ANALYTICAL FUNCTIONS.

II. SPHERICAL TRIGONOMETRY, WITH PRACTICAL AND NAUTICAL ASTRONOMY.

BY THE REV. JOHN WEST,

FORMERLY ASSISTANT TEACHER OF MATHEMATICS IN THE UNIVERSITY OF ST ANDREWS; THEREAFTER RECTOR OF ST THOMAS'S IN THE EAST, MORANT BAY, JAMAICA; AUTHOR OF A TREATISE ON THE ELEMENTS OF MATHEMATICS.

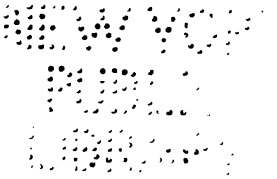
EDITED (AFTER THE AUTHOR'S DEATH) FROM HIS MSS.

BY THE LATE

SIR JOHN LESLIE,

PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF EDINBURGH.

ACCOMPANIED BY A MEMOIR OF THE LIFE AND WRITINGS OF THE AUTHOR, BY EDWARD SANG, F.R.S.E.



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LIFE

OF

JOHN WEST.

The man of science lives but in his study; and the history of his life, at least of that part of it which he himself deems worthy of publicity, is to be found in his writings.

Continually engaged in speculations for increasing the extent of human knowledge, or for ameliorating the state of human society, he judges that successes in these speculations are the prominent events of his life; and considers those circumstances which, however deeply they may affect himself, affect himself alone, as undeserving of the attention of mankind. Those family events which mark the different eras of human life, and shed an influence over all the actions of man, appear to

him to be connected with his proceedings as a Philosopher, only in so far as they impede or facilitate his researches; and were he to pen his own life, he would indicate those states of mind which were most congenial to scientific pursuits, and those trains of thought which led him to his various discoveries; he would delight to recount the difficulties which he had surmounted, and to describe those obstacles which had baffled his exertions; he would still write for the information and improvement of his fellow men. His biographer, in recording the difficulties which he encountered, and the encouragement which he received, ought steadily to keep in view the same objects.

And such records deserve, indeed, the attention of every lover of science. They teach us under what circumstances genius and virtue flourish; whether the lap of luxury or the thorny cradle of adversity be the proper nursing-place of genuine worth. Such records would inform the legislator, whether from the list of princely overseers, or from the crowd of neglected labourers in the fields of morality, the highest attainments may be expect-

ed. From them the Philanthropist could learn, whether the courtly education of our highly endowed universities, or the stern lessons of morality and public virtue, taught by an early experience of real life, be best fitted for awakening the ennobling sentiments of urbanity and independence.

JOHN WEST was the second son of the Reverend SAMUEL WEST, minister of Logie, in Fifeshire, and was born on the 10th April 1756. His father was ordained minister in May 1751, and, in August of the same year, married MARGARET MEIN, by whom he had seven children, JOHN being the third. Notwithstanding the smallness of the stipend, which, together with his salary, as presbytery clerk, could scarcely have amounted to £.85 *, Mr WEST contrived to give to his numerous family the best education which the country afforded. On his death, which happened about 1767, the care of the family devolved

* The succeeding incumbent, the Reverend Mr Bogie, states the stipend, in his statistical report of the parish, at £. 80, with a manse and glebe of four acres, and two acres of pasture grass. He quotes the then prices of provisions thus: Beef, 2d. to 2½d. ℔ lb. Mutton, 3d. A good fowl, 6d.

upon the widow, who, with that promptness and energy so frequently displayed by females in such situations, determined to follow the same course in their education.

In this laudable but arduous task, she was assisted by Dr ADAMSON, who kindly discharged the duties of presbytery clerk, in order that the salary attached to the office might be enjoyed by Mrs WEST.

The peculiarity of his situation seems to have had its proper effect on the honest and independent mind of young WEST, for, in 1775, we find that his exemplary conduct, and his proficiency in the exact sciences, had gained for him the prizes given by the Earl of KINNOUL, then Chancellor of the College of St Andrews, and induced Professor VILANT to commit the charge of his classes partly into the young man's hands. From that time till 1780, he assisted Professor VILANT, along with JAMES RENNIE, the celebrated engineer, or with Dr M'DONALD of Kembach.

During that time he was deprived of his mother, who died in January 1777; and thus, at the age of twenty-one years, he found himself charged with the care and maintenance of

his sisters, as his elder brother and younger brother had gone to Jamaica.

The interesting spectacle of a young family struggling with the world, seems to have engaged the Faculty in his behalf, since, in 1780, when he undertook the whole charge of the Mathematical Classes, through their interference his salary was raised from £.20 to two-thirds of the fees paid by the students, a change, however, which scarcely added ten pounds to his income. For four years longer he continued his labours at St Andrews, but finding some difficulty, no doubt, in supporting the family with such a slender income, although augmented by fees from private pupils, and disappointed in his hopes of college preferment, he came to the resolution of following his brother to Jamaica.

During his residence at St Andrews, his time must have been completely occupied by his studies, and by attention to his family duties: yet notwithstanding of his close retirement, he stood high in the opinion of all his acquaintances. The affections of the students he possessed in an eminent degree, not that affection which is sometimes manifested

to those who avoid the difficulties of the subject, but such as was due to a conscientious discharge of the duties of his office. Justly considering the soundness of the information conveyed to be of infinitely more importance than ornament in the language which conveyed it, he delivered his lectures in a plain but perspicuous language, which, if it did not attract the ear by the fineness of its elocution, improved the judgment by its accuracy. Nine years' practice in teaching could not fail to open his mind to new views of the subject, and of the method in which scientific knowledge can best be conveyed to the young mind. Brought into close and repeated contact with students, scarcely his juniors, anxious to imitate and excel their master, he had abundant opportunity of estimating the defects of the common system of education; and he was thus led to project his *Elements of Mathematics*, a work which displays originality as a mathematician, and skill as a teacher.

This work, which he published just before he departed for Jamaica, contains the germs of several since completed improvements in Geometry. He has placed the separation

between the first and second books of the Elements, so as to include in the latter all propositions relating to the quantity of surface; and thus, although he did not distinguish it by a new name, he laid the foundation for the theory of equivalence. Discarding the ancient definition of proportion, he introduced that which LESLIE adopted in his Elements of Geometry, and which the slightest consideration will show to be far better fitted for conveying to the mind a clear notion of ratio than was the old one. His definition was this, that "*of four magni-* "*tudes, the ratio of the first to the second is* "*said to be equal to the ratio of the third to* "*the fourth, when the first contains any part* "*whatever of the second, just as often as the* "*third contains the like part of the fourth,*" which LESLIE, with some change in the language, has transcribed thus: "*Four quantities* "*are said to be proportionable, when a part of* "*the first is contained in the second, as often as* "*a like part of the third is contained in the* "*fourth.*" The different properties of proportions are deduced with great ease from this definition; and indeed so centrally is it

placed, that almost any one can be proceeded to directly, and without the intervention of the others. The work, in other respects, bears evident marks of a master's hand,—but perhaps not of one well skilled in conciliating the good will (for his work I mean) of the great mass of teachers of the Mathematics. The unsolved problems which accompany each book present difficulties which the masters too often find troublesome; but which undoubtedly serve to sharpen the invention, and to excite the emulation of the students.

As a teacher, Mr West seems to have been eminently successful. The celebrity of his pupils, Leslie, Ivory and Chalmers, must have been to him matter of the highest gratulation. Their successes, under his care, may in fact have contributed to strengthen his passion for the exact sciences, and it is to be regretted, that any circumstances should have separated him from those, who, in a few years, would have appeared his equals in age, and particularly that he should have selected for his retreat a place so barren of all encouragement to scientific exertions as a slave colony. Trifles, however, often fix our de-

terminations in matters of the greatest moment, and it is easier to weigh the situation of our present affairs, than to estimate the remote influence of any line of conduct. There are few who would not have felt an interest in such a man as WEST,—few who would not have felt pleasure in assisting his struggle for independence. To the students he offered an example of high scientific attainments, reached, not through neglect, but in furtherance of his domestic duties. His fraternal piety spoke a language intelligible to all; yet even in the little circle of a university town, he seems to have met with some sordid minds incapable of estimating his real worth. Doubtless, on account of his high character and celebrity as a teacher, he became the tutor of a young nobleman, unfortunately without having made any agreement as to his payment; and at the close of the session he rejected the offered sum, choosing rather to be unpaid than to receive a pittance from those who, never having known the labour of earning, needed not to have been so niggard in the expenditure of money. There are who, under the colour of patroni-

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embarking on a new speculation, if not entirely a pleasurable sensation, is at least an engrossing one.

On landing at Kingston he engaged along with a countryman of his own in forming a circulating library; but the scheme proved unsuccessful. Having, however, devoted himself at the same time to teaching, he was more fortunate in his own department, and was soon appointed teacher of Mayson's free school at Savannah-la-Mar, in Westmoreland. He went thither with his two sisters, and gave great satisfaction to the people of that parish: so much so, that he was recommended by Fairlie Christie, Esq. a Member of the Assembly, to the notice of the Earl of Effingham. In the Governor, West found a steady and a zealous friend; and in that country, where to appearance there was the least chance of encouragement for the man of science, he first met with encouragement. Unfortunately, however, the Governor's field for patronage lay not in the scientific direction. He recommended West to take orders in the Church of England, offered him a letter to the Bishop of London, and promised

ing a young man of merit, take advantage of his zeal; and expect, as the price of their patronage, his unrecompensed labour, the appellative tutor to my Lord —— being, in their opinion, equivalent to a more substantial reward. The smallness of his receipts lead us to infer that this was not the only instance of such meanness, and in an evil hour for science, WEST buried his abilities in a society which, founded on the rupture of every moral principle, might check, but never could encourage, the developement of physical truths.

His younger brother having had occasion to be in England, had taken two of his sisters to Jamaica, and in 1784, JOHN WEST, with the two remaining sisters, took his departure for the same place. His feelings of regret at leaving a country dear to him from his early recollection, and as the scene of his youthful labours, must have been greatly diminished by the expectation of rejoining the other members of his family, for whom he had always felt the warmest affection. The young mind borrows strength from uncertainty, and the thrill of hope and of fear which is felt on

embarking on a new speculation, if not entirely a pleasurable sensation, is at least an engrossing one.

On landing at Kingston he engaged, along with a countryman of his own, in forming a circulating library; but the scheme proved unsuccessful. Having, however, devoted himself at the same time to teaching, he was more fortunate in his own department, and was soon appointed teacher of MANNING'S free school at Savannah-la-Mar, in Westmoreland. He went thither with his two sisters, and gave great satisfaction to the people of that parish; so much so, that he was recommended by FAIRLIE CHRISTIE, Esq. a Member of the Assembly, to the notice of the Earl of EFFINGHAM. In the Governor, WEST found a steady and a zealous friend; and in that country, where to appearance there was the least chance of encouragement for the man of science, he first met with encouragement. Unfortunately, however, the Governor's field for patronage lay not in the scientific direction. He recommended WEST to take orders in the Church of England, offered him a letter to the Bishop of LONDON, and promised

been placed in the parish church at Morant Bay, and thus conferred upon him an honour which had not perhaps been paid to any other clergyman in Jamaica. The calm pursuits of science estrange not the heart from the pleasures of social life, but tend rather to impress more deeply upon it the feelings of veneration for the Deity, and of love for mankind.

he had struck out for finding the longitude. He found the field, however, pre-occupied by MENDOZA, and retired from his attempt with an indistinct feeling that all had not been conducted fairly,—a feeling which had been occasioned by some of those circumstances, which, from their very triflingness, are calculated to give the clearest view of the actions and intentions of man. In London, he enjoyed with high relish the society of scientific men, and became so fond of it, that he seriously contemplated his permanent removal thither. The constraint which his free and honest mind must have felt, when living in the midst of slavery, and witnessing the debasement of his fellow men, and his complete separation from science and scientific men, strengthened his determination. He returned to Jamaica, however, and either found it impracticable, or had forgot his resolution, for he continued to reside at Morant Bay till his death, which happened on the 4th October 1817. Such was the high estimation in which his character was held, that the magistrates and vestry voted to his memory a monument, which, with a suitable inscription, has since

2. (1.) That fx, when $x = o$, may vanish, may be equal to some constant quantity, or may become infinite; if fx, for example, represent the 1st or 2d functions above specified, when $x = o$, $fx = o$; but if it denote the 3d or 4th, $fx = 1$; if the 5th or 6th, $fx = -c$, or $fx = \frac{c}{d}$; if the 7th or 8th, $fx = \infty$ (infinite).

3. (2.) That when a particular value is given to x, its function fx may, in that case, likewise vanish, or become infinite, or may be equal to some constant quantity; as, when $x = a$, in the last three functions above.

4. (3.) That when there is occasion to express different functions of the same variable quantity x, it may be done by Fx, φx, ψx, &c. or more simply by y, v, z, &c.

5. Again, $x^m + y^n$, $x^m y^n$, $a^b b^a$, $axy - by + c^c d$, &c. are functions of x, y; and, supposing x, y independent of one another, $f(x, y)$ is a general expression for such functions: And so on.

We propose in the following work to consider the functions of only a single variable quantity.

6. When the variable quantity x is increased to $x+i$, its function fx of course becomes $f(x+i)$. Now, we are first to shew how this latter function may be developed, so as to be expressed by a series in terms of the increment i, and different functions of the variable quantity x, derived from the primitive function fx.

Let $f(x+i)=fx+i\mathrm{P}$, then $\mathrm{P}=\frac{f(x+i)-fx}{i}$; and supposing i to vanish, this expression becomes $\mathrm{P}=\frac{o}{o}$, which, it is well known, is no evidence of P vanishing, but of its becoming of a determinate value independent of i: let that value be denoted by p, (which must therefore be a function of x,) and let $i\mathrm{Q}$ denote the part of P which vanishes with i; then, we have

$$\mathrm{P}=p+i\mathrm{Q};\ \text{therefore}\ \mathrm{Q}=\frac{\mathrm{P}-p}{i}.$$

This expression, again, when $i=o$, and consequently $\mathrm{P}=p$, becomes $\mathrm{Q}=\frac{o}{o}$: let q then, as before, be the

value of Q, when $i = o$, and iR the part of Q which vanishes with i; q is therefore a function of x without i, and $Q = q + iR$: And so on. Hence

$$f(x+i) = fx + iP = fx + ip + i^2Q = fx + ip + i^2q + i^3R$$

$$7. \qquad = fx + ip + i^2q + i^3r + i^4s + i^5t + \text{&c.}$$

where p, q, r, s, &c. are functions of x.

To illustrate this general process by particular examples.

8. (1.) Let $fx = \frac{1}{x}$, then $f(x+i) = \frac{1}{x+i}$.

Hence $iP = f(x+i) - fx = \frac{1}{x+i} - \frac{1}{x} = -\frac{i}{x(x+i)}$,

and $P = -\frac{1}{x(x+i)}$, consequently $p = -\frac{1}{x^2}$.

Again, $iQ = P - p = \frac{1}{x^2} - \frac{1}{x(x+i)} = \frac{i}{x^2(x+i)}$, and

$$Q = \frac{1}{x^2(x+i)}, \text{ consequently } q = \frac{1}{x^3}.$$

In like manner, $R = -\frac{1}{x^3(x+i)}$, and $r = -\frac{1}{x^4}$, &c. &c.

Thus, $\frac{1}{x+i} = \frac{1}{x} - \frac{i}{x^2} + \frac{i^2}{x^3} - \frac{i^3}{x^3(x+i)} =$

$$\frac{1}{x} - \frac{i}{x^2} + \frac{i^2}{x^3} - \frac{i^3}{x^4} + \frac{i^4}{x^5} - \frac{i^5}{x^6} + \text{&c.}$$

the series which results from actual division.

9. (2.) Let $fx = \sqrt{x}$ an irrational function of x, $f(x+i) = \sqrt{x+i}$.

Hence $iP = \sqrt{x+i} - \sqrt{x} = \frac{i}{\sqrt{x+i}+\sqrt{x}}$, and

$$P = \frac{1}{2\sqrt{x}}.$$

Again, $iQ = P - p = \frac{1}{\sqrt{x+i}+\sqrt{x}} - \frac{1}{2\sqrt{x}} =$

$$-\frac{\sqrt{x+i}-\sqrt{x}}{2\sqrt{x}(\sqrt{x+i}+\sqrt{x})} = -\frac{i}{2\sqrt{x}(\sqrt{x+i}+\sqrt{x})^2},$$

consequently $q = -\frac{1}{8x\sqrt{x}}$.

In like manner $R = \frac{\sqrt{x+i}+3\sqrt{x}}{8x\sqrt{x}(\sqrt{x+i}+\sqrt{x})^3}$, and

$$r = \frac{1}{16x^2\sqrt{x}}, \text{ \&c.}$$

Thus, $\sqrt{x+i} = \sqrt{x} + \frac{i}{2\sqrt{x}} - \frac{i^2}{8\sqrt{x^3}} + \frac{i^3}{16\sqrt{x^5}} -$

$$\frac{5i^4}{128\sqrt{x^7}} + \text{\&c.}$$

as may be found by extracting the square root.

10. In regard to the general series, $fx + ip + i^2q + i^3r + \text{\&c.}$ it is of importance to remark, that the

increment i may be taken so small, that any term shall be greater than the sum of all the following terms. For it is evident that i may be so small, that $f(x+i)$ shall be less than $2fx$; therefore fx may be greater than $f(x+i)-fx$, or $ip+i^2q+i^3r+i^4s+$ &c. In like manner, i may be so small, that P may be less than $2p$; therefore p may be greater than $P-p$, or iQ, and ip greater than i^2Q, or $i^2q+i^3r+i^4s+$ &c.

11. We are next to shew, that all the functions of x, which multiply the powers of i in the developement of $f(x+i)$, are so related to one another, and to the primitive function fx, that each may be derived from the preceding according to a certain law.

By substituting $i+o$ instead of i in the function $f(x+i)$ and its developement, we get

$$f(x+\overline{i+o})=fx+(i+o).p+(i+o)^2.q+(i+o)^3.r+(i+o)^4.s+ \text{ &c.}$$

$$\begin{aligned}
&= fx + ip + i^2.q + i^3.r + i^4.s + \text{ &c. } (=f(x+i)) \\
&\quad + op + 2io.q + 3i^2o.r + 4i^3o.s + \text{ &c.} \\
&\quad\quad + o^2.q + 3io^2.r + 6i^2o^2.s + \text{ &c.} \\
&\quad\quad\quad + o^3.r + 4io^3.s + \text{ &c.} \\
&\quad\quad\quad\quad + o^4.s + \text{ &c. &c.}
\end{aligned}$$

But $f(\overline{x+i}+o)$ is identically the same with $f(x+\overline{i+o})$; for we may consider o as the increment of $(x+i)$, as well as $(i+o)$ the increment of x; the same function, therefore, may be thus expressed,

$f(\overline{x+i}+o) = f(x+i) + o.\mathrm{A} + o^2.\mathrm{B} + o^3.\mathrm{C} + o^4.\mathrm{D} +$ &c.

where A, B, C, &c. are functions of $(x+i)$.

In order to compare this developement with the former, omitting the function $f(x+i)$, which is common to both, let its other functions A, B, C, &c. be expanded and multiplied by the corresponding power of o; we shall have,

$$\left.\begin{array}{r} o.\mathrm{A} \\ +o^2.\mathrm{B} \\ +o^3.\mathrm{C} \\ +o^4.\mathrm{D} \\ \text{\&c.} \end{array}\right\} = \begin{array}{l} o(p + i.p' + i^2.p'' + i^3.p''' + i^4.p^{\mathrm{IV}} +) \\ \quad + o^2.(q + i.q' + i^2.q'' + i^3.q''' +) \\ \quad\quad + o^3\,(r + i.r' + i^2.r'' +) \\ \quad\quad\quad + o^4(s + i.s' +) \\ \quad\quad\quad\quad \text{\&c.} \qquad \text{\&c.} \end{array}$$

Here, since the functions $f(x+i)$ and A are related to each other in the same way as fx and p, and fx is the leading function in the developement of $f(x+i)$, therefore p must be the leading function in the developement of A; and the succeeding functions may be denoted by p', p'', p''', &c. In like manner, q is the leading function in the developement of B; and the succeeding functions may be similarly expressed by q', q'', q''', &c.

Now, from the comparison of the co-efficients of the same quantities, in these different ways of expressing the same function, we obtain the following equations:

$$2q = p'.$$
$$2r = q'.$$
$$4s = r'.$$
$$5t = s'. \quad \&c.$$

regarding the second terms only of the several lines, the equality of which is sufficient to determine the law which pervades the whole.

But, the function p' being derived from p, q' from q, r' from r, &c. in the same manner, in order to express these derivations with more simplicity and uniformity, put $f'x$ for p, the function derived from fx; then $f''x$ will denote p', the function derived from (p, or) $f'x$; $f'''x$ that from $f''x$; $nf'''x$ that from $nf''x$; &c.

Thus,	$p =$	$f'x.$	Or, $p =$	$f'x.$
	$2q = p' =$	$f''x.$	$q =$	$\frac{1}{2} f''x.$
	$3r = q' =$	$\frac{1}{2} f'''x.$	$r =$	$\frac{1}{2.3} f'''x.$
	$4s = r' =$	$\frac{1}{2.3} f^{\text{IV}}x.$	$s =$	$\frac{1}{2.3.4} f^{\text{IV}}x.$
	$5t = s' =$	$\frac{1}{2.3.4} f^{\text{V}}x.$	$t =$	$\frac{1}{2.3.4.5} f^{\text{V}}x.$
	&c.		&c.	

By substituting these values of p, q, r, &c. in the developement of the function $f(x+i)$, we have,

12. $f(x+i) = fx+i.f'x+\frac{i^2}{2}f+x''\frac{i^3}{2.3}f'''x+\frac{i^4}{2.3.4}f^{IV}x$ + &c.

In which series, fx is the primitive, and $f'x$, $f''x$, $f'''x$, &c. its derivative functions; $f'x$ being the first derivative or *prime* function, $f''x$ the second derivative, &c.; and each of these derived from the preceding function in a similar manner.

If i be taken as a decrement of x, instead of an increment, as, in this case, the odd powers only of i are negative, we shall have,

13. $f(x-i) = fx-i.f'x+\frac{i^2}{2}f''x-\frac{i^3}{2.3}f'''x+\frac{i^4}{2.3.4}f^{IV}x$ — &c.

Hence, if in the former function, $f(x+i)$, and its developement, we substitute $x-i$ in place of x; and in the latter function, $f(x-i)$, and its developement, $x+i$, we get,

14. $$\begin{cases} fx=f(x-i)+i.f'(x-i)+\frac{i^2}{2}.f''(x-i)+\frac{i^3}{2.3}f'''(x-i)+\&c. \\ fx=f(x+i)-i.f'(x+i)+\frac{i^2}{2}.f''(x+i)-\frac{i^3}{2.3}f'''(x+i)+\&c. \end{cases}$$

Hence also, because $f(x+i) = f(\overline{x+\frac{1}{2}i}+\frac{1}{2}i)$,

and $fx = f(\overline{x+\frac{1}{2}i}-\frac{1}{2}i)$.

15. $f(x+i)+fx=2f\left(x+\frac{1}{2}i\right)+\frac{i^2}{2.2}f''\left(x+\frac{1}{2}i\right)$

$$+\frac{i^4}{2.3.4.2^3}f^{IV}\left(x+\frac{1}{2}i\right)+\text{ &c.}$$

16. $f(x+i)-fx=i.f'\left(x+\frac{1}{2}i\right)+\frac{i^3}{2.3.2^2}f'''\left(x+\frac{1}{2}i\right)$

$$+\frac{i^5}{2.3.4.5.2^4}f^{V}\left(x+\frac{1}{2}i\right)+\text{ &c.}$$

And, if y denote a function of x, its derivatives may, in like manner, be denoted by y', y'', y''', &c.; so that, when x becomes $(x+i)$, $y\,(=fx)$ will become,

$$y+i.y'+\frac{i^2}{2}.y''+\frac{i^3}{2.3}.y'''+\frac{i^4}{2.3.4}.y^{IV}+\text{ &c.}$$

Lastly, when $x=o$, let $f.$, $f'.$, $f''.$, &c. represent what the several functions, fx, $f'x$, $f''x$, &c. in that case become, then, if the said functions do not all vanish, nor any of them become infinite (2.), the developement above (12.) becomes,

$$fi=f.+i.f'.+\frac{i^2}{2}.f''.+\frac{i^3}{2.3}.f'''.+\frac{i^4}{2.3.4}.f^{IV}.+$$

But as i admits of any value, and is altogether independent of x, and as the functions $f.$, $f'.$, $f''.$, &c. (at least such of them as do not vanish,) are constant quan-

tities (2.), the equation must still subsist if we substitute x in place of i; therefore,

17. $fx=f.+x.f'.+\frac{x^2}{2}.f''.+\frac{x^3}{2.3}.f'''.+\frac{x^4}{2.3.4}.f^{\text{IV}}.+$ &c.

18. In prosecuting this subject, we have to shew,

(1.) The method of deriving the prime function, in all cases, from its primitive; and the uses of derivative functions. This is a direct method of Analysis, and is termed DERIVATION.

(2.) The method of obtaining the primitive, from its prime function; and some of the useful purposes to which it may be applied. This is the inverse of the preceding, and is termed INTEGRATION.

SECTION II.

OF FINDING THE PRIME FUNCTION.

FROM the preceding Theory it appears, that if we can by any means find the prime function of any function whatever of a single-variable quantity, we may, by a repetition of the same operation, obtain all its derivative functions, and, by consequence, all the terms of the series which result from the developement of the primitive, when the variable is increased or diminished by some constant quantity, (12. 13.). To shew how this may be effected, we shall now apply the Theory to these simple analytical functions of a single variable, viz.

x^n, a^x, log. x, sin. x, and cos. x,

and thence deduce certain well-known series.

I. FUNCTION x^n.

19. Let $fx = x^n$, then $f(x+i) = (x+i)^n$. Now, it is easy to prove, from the principles of Algebra, that

the first two terms of the n^{th} power of the binomial $(x+i)$ are x^n and nix^{n-1}, whether n be an integral or fractional number, positive or negative. We have then $f'x = nx^{n-1}$; hence, by a similar process, $f''x = n.\overline{n-1}.x^{n-2}$, $f'''x = n.n-1.n-2.x^{n-3}$, &c.; therefore, (12.)

$$(x+i)^n = x^n + n.x^{n-1}i + \frac{n.n-1}{2}.x^{n-2}i^2 + \frac{n.n-1.n-2}{2.3}.x^{n-3}i^3 + \text{ &c.}$$

Thus, to find the first derivative function of any given power of a variable quantity, we have this

RULE. Multiply the given index into the power of the variable, whose index is less by an unit than the given one, and the product is the prime function required.

EXAMPLE.

The prime function of x^5 is $5x^4$; of x, 1; of $\frac{1}{x^3}(=x^{-3})$, $-3x^{-4}$; &c. All the derivatives of x^5 are $5x^4$, $20x^3$, $60x^2$, $120x$, and 120.

II. FUNCTION a^x.

20. Let $fx = a^x$, then $f(x+i) = a^{x+i} = a^x.a_i =$ (suppose) $a^x.(1+b)^i$.

logs. of one system may be obtained from the corresponding logs. of another. Common logs., for example, may be deduced from hyperbolic, by multiplying each of the latter into m, the reciprocal of the hyp. log. of the base of the former. And this multiple m, (= .43429, &c.) is termed the Modulus of the common system.

But with respect to hyp. logs., (which alone shall be regarded in what follows,) we found above that A, the hyp. log. of a, is expressed by $b-\frac{1}{2}b^2+\frac{1}{3}b^3-\frac{1}{4}b^4+$&c. (20.), that is, by $(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+$&c. therefore also, by substituting $\overline{1+y}$ in place of a,

24. $\text{Log.}\,(1+y)=y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\frac{1}{5}y^5-$&c.

And, because, if y be negative, all the terms are negative.

25. $\text{Log.}\frac{1+y}{1-y}=2(y+\frac{1}{3}y^3+\frac{1}{5}y^5+\frac{1}{7}y^7+$&c.)

Also, since $\log.(1+y)=-l.\frac{1}{1+y}=-l.\left(1-\frac{y}{1+y}\right)$,

26. $\text{Log.}(1+y)=\frac{y}{1+y}+\frac{1}{2}\cdot\frac{y^2}{(1+y)^2}+\frac{1}{3}\cdot\frac{y^3}{(1+y)^3}+\frac{1}{4}\cdot\frac{y^4}{(1+y)^4}+$&c.

Lastly, because $\log. y=l.(1+y)-l.\left(1+\frac{1}{y}\right)$.

27. $\text{Log.}\,y=\frac{y-y^{-1}}{1}+\frac{y^2-y^{-2}}{2}+\frac{y^3-y^{-3}}{3}+\frac{y^4-y^{-4}}{4}+$&c.

III. FUNCTION, LOG. x.

28. Let $fx = l.x$, then $f(x+i) = l.(x+i) = l.x + l.\frac{x+i}{x} = l.x + l.\left(1+\frac{i}{x}\right) = l.x + i\frac{1}{x} - \frac{i^2}{2}\frac{1}{x^2} +$ &c. (24.). Hence $f'x = \frac{1}{x}$, $f''x = -\frac{1}{x^2}$, $f'''x = \frac{2}{x^3}$, $f^{IV}x = -\frac{2.3}{x^4}$, &c.

Thus, the prime function of the hyp. log. x is $\frac{1}{x}$, and in any other system of which m is the modulus, the prime function of log. x is $\frac{m}{x}$.

RULE. The prime function of the log. of a variable quantity is the product of the reciprocal of that quantity and the modulus of the system.

IV. FUNCTION, SIN. x.

29. Let $fx = \text{sin. } x$, then $f(x+i) = \text{sin. } (x+i) =$ (suppose) $\text{sin. } x + i\text{P}$; therefore $\text{P} = \frac{\text{sin. } (x+i) - \text{sin. } x}{i} = \frac{2 \text{ sin. } \frac{1}{2}i.\text{cos. } (x+\frac{1}{2}i)}{i}$, which (since the arc i and

its chord 2 sin. $\frac{1}{2}i$ are ultimately equal) becomes, when i vanishes, $p = \cos. x$.

RULE. The prime function of sin. x is cos. x.

V. FUNCTION, COS. x.

30. Here $fx = \cos. x$, and $f(x+i) = \cos. (x+i)$ $= \cos. x + i\mathrm{P}$; therefore $\mathrm{P} = \frac{\cos. (x+i) - \cos. x}{i}$ $= -\frac{2 \sin. \frac{1}{2} i \,.\, \sin. (x + \frac{1}{2} i)}{i}$, which, when i vanishes, becomes $p = -\sin. x$.

RULE. The prime function of cos. x is $-$ sin. x.

Hence, if $fx = \sin. x$, its derivatives are,

$+\cos. x, -\sin. x, -\cos. x, +\sin. x, +\cos. x, -$ &c.

And, if $fx = \cos. x$, its derivatives are,

$-\sin. x, -\cos. x, +\sin. x, +\cos. x, -\sin. x, -$ &c.

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31. The prime function, then, of each of these simple analytical functions of x being known, we find that

of any compound algebraical expression, by taking the prime functions of its several parts. Thus, the prime function of $\frac{x^3}{a} - by + c^2 + \frac{b^2c}{x}$ (where y represents any function of x) is $\frac{3x^2}{a} - by' - \frac{b^2c}{x^2}$. Hence, we observe, that if any function be multiplied by a given quantity, the same multiple appears in its prime function; but if the given quantity be added to the former, it disappears in the latter.

EXAMPLES.

Primitives, ax^2, $3x^4$, x, $-2bx$, $-2b$, $a^2x - bx^2$.
1st Derivatives, $2ax$, $12x^3$, 1, $-2b$, 0, $a^2 - 2bx$.

Prim. $a-x$, $(a-x)^2$, $\sqrt{x}$, $\sqrt{px}$, $\frac{1}{x}$, $-\frac{1}{x^2}$, $\frac{1}{x^n}$.

1st Der. -1, $-2a+2x$, $\frac{1}{2\sqrt{x}}$, $\frac{\sqrt{p}}{2\sqrt{x}}$, $-\frac{1}{x^2}$, $+\frac{2}{x^3}$, $-\frac{n}{x^{n+1}}$.

32. As the prime function of x, which is 1, disappears in the first derivative of every function of x, the young analyst, without particular attention, may be apt to misapply the preceding rules. He may conclude, for instance, that, because the prime function of $l.x$ is $\frac{1}{x}$, that of $l.nx$ must be $\frac{1}{nx}$. Whereas, if he consider

that $l.nx = l.n + l.x$, and $l.x^n = nl.x$, and attend to the above observation (31.), he will see that $\frac{1}{x}$ must be the prime function of the former, and $\frac{n}{x}$ the prime function of the latter.

But the rules will be rendered more explicit, as well as more general, in the next Section.

SECTION III.

TO FIND THE PRIME FUNCTION, WHEN THE VARIABLE QUANTITY IS ITSELF A FUNCTION OF SOME OTHER VARIABLE.

33. SUPPOSING y, v, z, &c., as before, to denote functions of x, their prime functions, we have said, may be denoted by y', v', z', &c. But it remains to be shewn how any function whatever of y, v, z, &c. may be expressed, and its prime function determined.

Now fy, fz, &c. agreeably to the notation already adopted, may express any function whatever of y, z,

&c.; and, considering fy, fz as primitives, their first derivatives will be found to be $y'.f'y$ and $z'.f'z$.

For, when x becomes $x + i$, y becomes

$$y + i.y' + \frac{i^2}{2} \cdot y'' + \frac{i^3}{2.3} \cdot y''' + \frac{i^4}{2.3.4} \cdot y^{iv} + \&c.$$

$$=, \text{ we may suppose, to } y + o.$$

But, when y becomes $y + o$, fy becomes

$$f(y + o) = fy + o.f'y + \frac{o^2}{2} f''y + \frac{o^3}{2.3} \cdot f'''y + \&c.$$

therefore, when x becomes $x + i$, fy becomes $fy + o.f'y +$ &c. or (substituting for o its value above) $fy + i.y'f'y +$ &c. Hence $y'f'y$ (the multiplier of i) is the prime function of fy, when y is a function of x.

34. When x is considered as the *original* variable, of which all others are functions, then the prime function of fx, any function whatever of x, is denoted simply by $f'x$. But, if x is to be considered as itself a function of some other variable, the prime function of fx, it appears, must, in this case, be expressed by $x'.f'x$.

We may therefore consider x as the function of some other variable, and render the preceding rules for finding the prime function more general, if we only mul-

tiply by x' each of the prime functions found by these rules.

35. Again, let $f(y, z)$ denote any compound function of y and z, (which are themselves functions of x,) and its prime function will be expressed by $y'.f'(y) + z'.f'(z)$, where $f'(y)$ is used to denote the prime function of $f(y, z)$, when z is considered as a constant quantity, and $f'(z)$ to denote the same, when y is considered constant. For, if z be constant, $y'.f'(y)$ is the multiple of i; and if y be constant, $z'.f'(z)$ is the multiple of i; therefore, when both y and z are variable, $y'.f'(y) + z'.f'(z)$ must be the multiple of i, or prime function of $f(y, z)$.

To give a particular example: The prime function of yz, supposing y constant, is yz'; and, supposing z constant, it is zy'; therefore, both being variable, it is $yz' + zy'$. For, when x becomes $x + i$, y becomes $y + i.y' +$ &c., z becomes $z + i.z' +$ &c. and the given function becomes $yz + i(yz' + zy') +$ &c. where the multiple of i is the prime function.

36. Hence these general rules for finding the prime function, when the variable quantities of the primitive are all functions of one and the same variable.

PRIMITIVE.	PRIME FUNCTION.
1. x^n.	$nx^{n-1} \cdot x'$.
2. a^x.	$l \cdot a \cdot a^x \cdot x'$.
3. $l \cdot x^n$.	$\frac{nx'}{x} \left(= \frac{nx^{n-1}x'}{x^n} \right)$.
4. sin. x.	cos. $x \cdot x'$.
5. cos. x.	$-$ sin. $x \cdot x'$.
6. xy.	$xy' + yx' = xy\left(\frac{x'}{x} + \frac{y'}{y}\right)$.
7. xyz.	$xyz' + xzy' + yzx' = xyz\left(\frac{x'}{x} + \frac{y'}{y} + \frac{z'}{z}\right)$.
8. $\frac{x}{y}$.	$\frac{yx' - xy'}{y^2} = \frac{x}{y}\left(\frac{x'}{x} - \frac{y'}{y}\right)$.

EXAMPLES

IN ILLUSTRATION OF THE PRECEDING RULES.

37. PRIMITIVES.	PRIME FUNCTIONS.
1. $ax^3 - bx^2 + cx - d$.	$3ax^2x' - 2bxx' + cx'$.
2. $a + b\sqrt{x} - \frac{c}{x}$.	$\frac{bx'}{2\sqrt{x}} + \frac{cx'}{x^2}$.
3. $(a + bx^n)^m$.	$mnb \cdot (a + bx^n)^{m-1} \cdot x^{n-1} \cdot x'$.
4. $\sqrt{a^2 - x^2}$.	$-\frac{xx'}{\sqrt{a^2 - x^2}}$.
5. $x\sqrt{a^2 - x^2}$.	$x'\sqrt{a^2 - x^2} - \frac{x^2x'}{\sqrt{a^2 - x^2}}$.

PRIMITIVES.	PRIME FUNCTIONS.
6. $\frac{x}{a-x}$.	$ax'(a-x)^{-2}$.
7. $\frac{c}{a-x}$.	$-cx'(a-x)^{-2}$.
8. $\frac{1+x}{1-x}$.	$2x'(1-x)^{-2}$.
9. $\sqrt{1+x}+\sqrt{a-x}$.	$\frac{1}{2}x'(1+x)^{-\frac{1}{2}}-\frac{3}{2}x'(a-x)^{\frac{1}{2}}$.
10. $\sqrt{a^3-b^2x+cx^2}$.	$\frac{(2cx-b^2)x'}{\sqrt{a^3-b^2x+cx^2}}$.
11. $\frac{x}{\sqrt{a^2+x^2}}$.	$ax'(a^2+x^2)^{-\frac{1}{2}}$.
12. $\sqrt{(a^2+x^2)(a^2-x^2)}$.	$x'(a^4+a^2x^2-4x^4)(a^2-x^2)^{-\frac{1}{2}}$.
13. $\frac{(1+x^2)^{\frac{3}{2}}}{3x^3}$.	$(1+x^2)^{\frac{1}{2}}x'-(1+x^2)^{\frac{3}{2}}\cdot\frac{x'}{3x^3}$.

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14. $\frac{x}{a+x}$.	$\frac{ax'}{a+x}$
15. $\frac{ax}{a+x}+\frac{ax}{b-x}$.	$\frac{ax'}{a+x}-\frac{ax'}{b-x}$.
16. $\sqrt{a^2+x^2}-a\log\frac{a+\sqrt{a^2+x^2}}{x}$.	$\frac{a^2x'}{x\sqrt{a^2+x^2}}$.
17. x^{n-2}.	$(n-2)x^{n-3}\cdot\frac{x'}{x}$.
18. $\frac{1}{2}\left(\sin^{-1}x-\frac{x}{1+x^2}\right)$.	$\frac{x'}{x\sqrt{1-x^2}}$.

PRIMITIVES.	PRIME FUNCTIONS.
19. $l.l.x.$	$\frac{(l.x)'}{l.x} = \frac{x'}{x.l.x}.$
20. $l\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)^{\frac{1}{2}} = l(x+\sqrt{1+x^2}.$	$\frac{x'}{\sqrt{1+x^2}}$
21. $\frac{1}{\sqrt{-1}} l.(x\sqrt{-1}+\sqrt{1-x^2}).$	$\frac{x'}{\sqrt{1-x^2}}.$
22. $y^z.$	$y^z\left(z.\frac{y'}{y}+z'.l.y\right).$
23. $a^{x^2}.$	$l.a.l.n.a^{x^2}.n^x.x'.$

Of the arc z, to the radius 1, let s, c, t, v and f be the sine, cosine, tangent, versed sine and secant; then

24. $z' = \frac{s'}{\sqrt{1-s^2}} = -\frac{c'}{\sqrt{1-c^2}} = \frac{t'}{1+t^2} = \frac{v'}{\sqrt{2v-v^2}} = \frac{f'}{f\sqrt{f^2-1}}.$

Hence the following eqnations, putting $r =$ radius.

25. $rs' = cz'$, $rc' = -sz'$, $rv' = sz'$, $r^2t' = f^2z'$, and $r^2f' = ftz'$.

26. If $s = 2y\sqrt{1-y^2}$, then $z' = \frac{2y'}{\sqrt{1-y^2}}$, or $\frac{1}{2}z' = \frac{y'}{\sqrt{1-y^2}}$.

27. If $t = \frac{x}{y}$, then $z' = \frac{yx' - xy'}{x^2+y^2}$.

28. If $y = l.\sec. z$, then $y' = tz'$.

29. $(a \sin. nz)' = an. \cos. nz.z'.$; $(a \cos. nz)' = -na. \sin. nz.z'.$

$$\ldots + i\left(\overline{x-a}^{\frac{1}{2}} + \tfrac{1}{2}x-b\,\overline{x-a}^{-\frac{1}{2}}\right) + \frac{i^2}{2}\cdot$$

$$\ldots + \frac{i^3}{2.3}\left(-\tfrac{1}{4}x-a^{-\frac{3}{2}} + \tfrac{3}{8}x-b\,.\,x-a^{-\frac{5}{2}}\right) + \&c.$$

where, if we put the function $(x-a)^{\frac{1}{2}} = o$, or $x = a$, the primitive vanishes, and all its derivatives become infinite. In this particular case, therefore, the series fails. Nevertheless the said series ceases not to be the true developement of $f(x+i)$, as long as x is greater than a. For, putting $x-a = v$, the above series becomes

$$\ldots + i\sqrt{v} - \frac{i}{2\sqrt{v}}x-b + \frac{i^2}{2\sqrt{v}} - \frac{i^2}{8\sqrt{v^3}}(x-b) - \frac{i^3}{8\sqrt{v^3}} + \&c.$$

$$= \ldots x - b + i\sqrt{v} + \frac{i}{2\sqrt{v}} - \frac{i^2}{8\sqrt{v^3}} + \frac{i^3}{16\sqrt{v^3}} - \&c.)$$

$$= \ldots x - b + i\,.\,\sqrt{v+i} = (x+i-b)\,.\,\sqrt{x+i-a} = f(x+i).$$

Here, by reducing the series to its equivalent expression $(x+i-b)\,.\,\sqrt{v+i}$, and then supposing v to vanish, (that is, assuming $x = a$) we get $(a+i-b)\sqrt{i}$ for the true developement in that particular case.

Thus also, $\frac{bx}{a-x} + x$, and $b\,.\,l\,.\,(x-a)^3$ are functions which, with all their derivatives, become infinite when $x = a$. And $(a-x)^{\frac{1}{2}} + cx^n$ is a function, whose ... preceding derivatives become infinite when ...

$x = a$; for the index $\frac{7}{2}$ being diminished by 1 in the first and in every succeeding derivative, must become negative in the 4th.

39. Again, if any one of the functions fx, $f'x$, $f''x$, &c. contain any quantity which has not a negative index, either in the said function, or in any of its derivatives, and that quantity be made to vanish, by putting $x = a$, the containing function, and one or more of its derivatives may, in that case, vanish; but all these functions will not vanish, nor will any of them become infinite.

Taking the same example as above, $fx = (x-a)^{\frac{1}{2}} . (x-b)$, and supposing $x - b = o$, or $x = b$, it is evident that only the primitive in that case vanishes.

But $(x-a)^{\frac{1}{2}} . (x-b)^2$ is a function, in which, on the same supposition, the prime function, as well as the primitive, but no other derivative, vanishes.

Thus, it appears, that, so long as x continues indeterminate, the series $fx + i.f'x + \frac{i^2}{2} . f''x + \frac{i^3}{2.3} . f''' +$ &c. is the true developement of $f(x+i)$; and that it only ceases to be so, when such a particular value is assigned to x, as to render any one term of the series (and consequently all its derivatives) infinite.

PRIMITIVES.	PRIME FUNCTIONS.
6. $\frac{x}{a-x}$.	$ax'(a-x)^{-2}$.
7. $\frac{c}{a-x}$.	$-cx'(a-x)^{-2}$.
8. $\frac{1+x}{1-x}$.	$2x'(1-x)^{-2}$.
9. $\sqrt{1+x}+(a-x)^3$.	$\frac{1}{2}x'(1+x)^{-\frac{1}{2}}-3x'(a-x)^2$.
10. $\sqrt{a^3-b^2x+cx^2}$.	$\frac{(2cx-b^2).x'}{\sqrt{a^3-b^2x+cx^2}}$.
11. $\frac{x}{a\sqrt{a^2+x^2}}$.	$ax'.(a^2+x^2)^{-\frac{1}{2}}$.
12. $x(a^2+x^2)(a^2-x^2)^{\frac{1}{2}}$.	$x'(a^4+a^2x^2-4x^4)(a^2-x^2)^{-\frac{1}{2}}$.
13. $\frac{(1+x^2)^{\frac{1}{2}}}{3x}$.	$(1+x^2)^{\frac{1}{2}}.x'-(1+x^2)^{\frac{1}{2}}.\frac{x'}{3x^2}$.

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14. $l.(ab+bx)$.	$\frac{x'}{a+x}$.
15. $l.\overline{(a+x)}^m.\overline{b-x}^n$.	$\frac{mx'}{a+x}-\frac{nx'}{b-x}$.
16. $l.\frac{x}{\sqrt{a^2+x^2}}=l.x-\frac{1}{2}l.(a^2+x^2)$.	$\frac{a^2x'}{x.(a^2+x^2)}$.
17. $(l.x)^n$.	$n(l.x)^{n-1}.\frac{x'}{x}$.
18. $l.\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)=l.\frac{1+\sqrt{1-x^2}}{x}$.	$-\frac{x'}{x\sqrt{1-x^2}}$.

PRIMITIVES.	PRIME FUNCTIONS.
19. $l.l.x.$	$\frac{(l.x)'}{l.x} = \frac{x'}{x.l.x}$.
20. $l\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)^{\frac{1}{2}} = l(x+\sqrt{1+x^2})$.	$\frac{x'}{\sqrt{1+x^2}}$.
21. $\frac{1}{\sqrt{-1}} l.(x\sqrt{-1}+\sqrt{1-x^2})$.	$\frac{x'}{\sqrt{1-x^2}}$.
22. y^z.	$y^z\left(z.\frac{y'}{y}+z'.l.y\right)$.
23. a^{n^x}.	$l.a.l.n.a^{n^x}.n^x.x'$.

Of the arc z, to the radius 1, let s, c, t, v and f be the sine, cosine, tangent, versed sine and secant; then

24. $z' = \frac{s'}{\sqrt{1-s^2}} = -\frac{c'}{\sqrt{1-c^2}} = \frac{t'}{1+t^2} = \frac{v'}{\sqrt{2v-v^2}} = \frac{f'}{f\sqrt{f^2-1}}$.

Hence the following eqnations, putting $r =$ radius.

25. $rs' = cz'$, $rc' = -sz'$, $rv' = sz'$, $r^2t' = f^2z'$, and $r^2f' = ftz'$.

26. If $s = 2y\sqrt{1-y^2}$, then $z' = \frac{2y'}{\sqrt{1-y^2}}$, or $\frac{1}{2}z' = \frac{y'}{\sqrt{1-y^2}}$.

27. If $t = \frac{x}{y}$, then $z' = \frac{yx' - xy'}{x^2+y^2}$.

28. If $y = l.\sec.z$, then $y' = tz'$.

29. $(a\sin.nz)' = an.\cos.nz.z'.$; $(a\cos.nz)' = -na.\sin.nz.z'$.

30. $(a \sin. nz)^{R}$, or the derivative function of $(a \sin. nz)$ expressed by R $(=r)$, is

$$a n^r . \cos. nz . z', \text{ if } r = 1, 5, 9, 13, \&c.$$
$$-a n^r . \sin. nz . z', \text{ if } r = 2, 6, 10, 14, \&c.$$
$$-a n^r . \cos. nz . z', \text{ if } r = 3, 7, 11, 15, \&c.$$
$$a n^r . \sin. nz . z', \text{ if } r = 4, 8, 12, 16, \&c.$$

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31. $x^m y^n$. $\quad mx^{m-1}y^n x' + ny^{n-1}x^m y'$.

32. $\dfrac{x-y}{x}$. $\quad \dfrac{yx' - xy'}{x^2}$.

33. $\sqrt{x^2+y^2}$. $\quad (xx' + yy') . (x^2 + y^2)^{-\frac{1}{2}}$.

34. $x\sqrt{a^2+y^2}$. $\quad x'\sqrt{a^2+y^2} + xyy'(a^2+y^2)^{-\frac{1}{2}}$.

35. $\dfrac{ay}{\sqrt{x^2+y^2}}$. $\quad (ax^2y' - ayxx') . (x^2 + y^2)^{-\frac{1}{2}}$.

36. $x^4y - ax^3y^2 + x^2y^3 - cxy^4$; $\quad x^4y' - 2ax^3yy' + 3x^2y^2y' - 4cxy^3y' + 4yx^3x' - 3ay^2x^2x' + 2y^3xx' - cy^4x'$.

SECTION IV.

CASES IN WHICH THE SERIES OF DERIVATIVE FUNCTIONS FAILS.

38. It was observed before, that, when we assign a particular value to x, its function fx may, in that case, vanish, become infinite, or be equal to some constant quantity.

We now observe farther, that, if fx, $f'x$, $f''x$, &c. contain any function of x with a negative index, and that function vanish when a particular value is assigned to x, not only the containing function, but all its derivatives do, in that case, become infinite. In such cases, therefore, the developement by these derivative functions must necessarily fail.

If, for example, $fx = (x-a)^{\frac{1}{2}}(x-b)$, then

$f'x = (x-a)^{\frac{1}{2}} + \frac{1}{2}(x-b)$

$f''x = (x-a)^{-\frac{1}{2}} - \frac{1}{4}(x-b)$

$f'''x =$

&c.

And $f(x+i)=(x-a)^{\frac{1}{2}}.(x-b)+i.\left(\overline{x-a}^{\frac{1}{2}}+\frac{1}{2}(x-b).\overline{x-a}^{-\frac{1}{2}}\right)+\frac{i^2}{2}.$

$\left(\overline{x-a}^{-\frac{1}{2}}-\frac{1}{4}(x-b).(x-a)^{-\frac{3}{2}}.\right)+\frac{i^3}{2.3}\left(-\frac{3}{4}(x-a)^{-\frac{3}{2}}+\frac{3}{8}(x-b).(x-a)^{-\frac{5}{2}}\right)+$ &c.

where, if we put the function $(x-a)^{\frac{1}{2}}=o$, or $x=a$, the primitive vanishes, and all its derivatives become infinite. In this particular case, therefore, the series fails. Nevertheless the said series ceases not to be the true developement of $f(x+i)$, as long as x is greater than a. For, putting $x-a=v$, the above series becomes

$$(x-b)\sqrt{v}+i\sqrt{v}+\frac{i}{2\sqrt{v}}(x-b)+\frac{i^2}{2\sqrt{v}}-\frac{i^2}{8\sqrt{v^3}}(x-b)-\frac{i^3}{8\sqrt{v^3}}+\text{ &c.}$$

$$=(x-b+i).\left(\sqrt{v}+\frac{i}{2\sqrt{v}}-\frac{i^2}{8\sqrt{v^3}}+\frac{i^3}{16\sqrt{v^3}}-\text{ &c.}\right)$$

$$=(x-b+i).\sqrt{v+i}=(x+i-b).\sqrt{x+i-a}=f(x+i).$$

Here, by reducing the series to its equivalent expression $(x+i-b).\sqrt{v+i}$, and then supposing v to vanish, (that is, assuming $x=a$) we get $(a+i-b)\sqrt{i}$, for the true developement in that particular case.

Thus, also, $\frac{bx}{a-x}+x$, and $b.l.(x-a)^3$ are functions, which, with all their derivatives, become infinite when $x=a$: And $(a-x)^{\frac{1}{2}}+cx^3$ is a function, whose 4th and all succeeding derivatives become infinite when

$x = a$; for the index $\frac{7}{2}$ being diminished by 1 in the first and in every succeeding derivative, must become negative in the 4th.

39. Again, if any one of the functions fx, $f'x$, $f''x$, &c. contain any quantity which has not a negative index, either in the said function, or in any of its derivatives, and that quantity be made to vanish, by putting $x = a$, the containing function, and one or more of its derivatives may, in that case, vanish; but all these functions will not vanish, nor will any of them become infinite.

Taking the same example as above, $fx = (x - a)^{\frac{1}{2}} . (x - b)$, and supposing $x - b = o$, or $x = b$, it is evident that only the primitive in that case vanishes.

But $(x - a)^{\frac{1}{2}} . (x - b)^{2}$ is a function, in which, on the same supposition, the prime function, as well as the primitive, but no other derivative, vanishes.

Thus, it appears, that, so long as x continues indeterminate, the series $fx + i.f'x + \frac{i^2}{2} . f''x + \frac{i^3}{2.3} . f''' +$ is the true developement of $f(x + i)$; and that it only ceases to be so, when such a particular value is assigned to x, as to render any one term of the series (and consequently all its derivatives) infinite.

SECTION V.

APPLICATION OF THE GENERAL THEOREMS IN SECTION I.

40. BEING thus enabled to find the prime function of any analytical function whatever, we may, by a simple application of the preceding general theorems to particular cases, obtain a variety of useful series', as will appear from the following examples.

EXAMPLE 1.

From the sine and cosine of x, to find those of $\overline{x+a}$.

We have $f(x+a)=fx+a.f'x+\frac{a^2}{2}f''x+\frac{a^3}{2.3}.f'''x+$ &c.

And, if $fx=\sin. x$, then $f'x=\cos. x$, $f''x=-\sin. x$, &c.

Or, if $fx=\cos. x$, $f'x=-\sin. x$, $f''x=-\cos. x$, &c.

Hence,

$$\sin.(x+a)=\sin. x+a.\cos. x-\frac{a^2}{2}\sin. x-\frac{a^3}{2.3}\cos. x+\&c.$$

$$\cos.(x+a)=\cos. x-a.\sin. x-\frac{a^2}{2}\cos. x+\frac{a^3}{2.3}\sin. x+\&c.$$

EXAMPLE 2.

From the hyp. log. x, to find that of $\overline{x+a}$.

Since (16.) $fx+a)=fx+\frac{2a}{2}f'(x+\frac{1}{2}a)+\frac{2a^2}{2.3.2^2}$ $f'''.(x+\frac{1}{2}a)$ + &c. if $f(x+a)=l.(x+a)$, then

$f'(x+\frac{1}{2}a)=\frac{2}{2x+a}$; $f''(x+\frac{1}{2}a)=-\frac{4}{(2x+a)^2}$;

$f'''(x+\frac{1}{2}a)=\frac{2.8}{(2x+a)^3}$; $f^{IV}(x+\frac{1}{2}a)=-\frac{2.3.16}{(2x+a)^4}$;

$f\ (x+\frac{1}{2}a)=\frac{2.3.4.32}{(2x+a)^5}$, &c.; therefore,

$$l.(x+a)=l.x+\frac{2a}{2x+a}+\frac{2a^3}{3(2x+a)^3}+\frac{2a^5}{5(2x+a)^5}+\frac{2a^7}{7(2x+a)^7}+\&c.$$

EXAMPLE 3.

To find the sine and tangent of an arc in terms of the arch.

Let x be the arc, of which s, c, t are the sine, cosine and tangent; then since (17.)

$$fx=f.+xf'.+\frac{x^2}{2}.f''.+\frac{x^3}{2.3}f'''.+\frac{x^4}{2.3.4}f^{IV}.+\&c.$$

and $fx=s$, $f'x=c$, $f''x=-s$, $f'''x=-c$, $f^{IV}x=+s$, &c. when $x=o$, fx and all the even derivative functions

vanish; but the odd derivatives become each $= \pm c = \pm 1$, alternately affirmative and negative. Hence,

$$s = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2 \ldots . 7} + \frac{x^9}{2 \ldots . . . 9} - \&c.$$

Next, let $fx = t$, then (supposing $x' = 1$ as before) $f'x = t'$. But (37. 25.) $t' = 1 + t^2$; therefore,

$f'x = 1 + t^2$. $\quad f''x = 2t + 2t^2$.

$f'''x = 2 + 8t^2 + 6t^4$. $\quad f^{IV}x = 16t + 40t^3 + 24t^5$.

$f^{V}x = 16 + 136t^2 + 240t^4 + 120t^6$.

$f^{VI}x = 272t + 1232t^3 + 1680t^5 + \&c.$

$f^{VII}x = 272 + 3968t^2 + \&c.$

Now, when $x = o$, $t = o$, so that all the even derivative functions in this case also vanish, and the odd derivatives become 1, 2, 16, 272, &c. Consequently,

$$t = x + \frac{2x^3}{2.3} + \frac{16x^5}{2.3.4.5} + \frac{272x^7}{2 \ldots . 7} + \&c.$$

EXAMPLE 4.

To find the m^{th} power of the polynome $a + bx + cx^2 + dx^3 + \&c.$

Suppose $fx = (a + bx + cx^2 + dx^3 + \&c.)^m = y^m$.

Then, since $fx = f. + xf'. + \frac{x^2}{2} f'' + \frac{x^3}{2.3} f''' + \&c.$

and $fx = y^m$, $f'x = my^{m-1}y'$, $f''x = m.m-1.y^{m-2}.y'^2 + my^{m-1}.y''$, $f'''x = m.m-1.m-2.y^{m-3}.y'^3 + 3m.m-1.y^{m-2}.y'y'' + my^{m-1}.y'''$, &c.

Also, $y = a + bx + cx^2 + dx^3 + ex^4 + fx^5 +$ &c.

$y' = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4 +$ &c.

$y'' = 2c + 2.3dx + 3.4ex^2 + 4.5fx^3 +$ &c.

$y^{IV} = 2.3d + 2.3.4x + 3.4.5fx^2 +$ &c.

&c. &c.

When $x = o$, $y = a$, $y' = b$, $y'' = 2c$, $y''' = 2.3d$, $y^{IV} = 2.3.4e$, &c. Hence $f. = a$, $f'. = ma^{m-1}b$, $f''. = m.m-1.a^{m-2}.b^2 + 2ma^{m-1}c$, $f'''. = m.m-1.m-2.a^{m-3}b^3 + 2.3m.m-1.a^{m-2}bc + 2.3ma^{m-1}d$, &c.

And,

$$y^m = a^m + ma^{m-1}bx + \left(\frac{m.m-1}{2}a^{m-2}b^2 + ma^{m-1}c\right)x^2 + \text{&c.}$$

SECTION VI.

OF DERIVATIVE EQUATIONS.

41. When the relation between two variable quantities, x, y, is determined by an equation expressed in general terms by

$$F(x, y) = o,$$

it is evident that, by the solution of this equation, y may be found in terms of x; y is therefore a function of x, and may be denoted by fx. Hence,

$$F(x, fx) = o.$$

But $F(x, fx)$ being a function of x only, may be denoted by φx; then $\varphi x = o$, whatever be the value of x. Hence also $\varphi(x + i) = o$, for $x + i$ may be considered as a value of x; thus,

$$o = \varphi x + i \,.\, \varphi' x + \frac{i^2}{2} \,.\, \varphi'' x + \frac{i^3}{2.3} \varphi''' x + \text{\&c.};$$

where, since $\varphi x = o$, and i has a real value independent of x, the equation cannot subsist unless at the same time $\varphi' x = o$, $\varphi'' x = o$, &c.

From any primitive equation, therefore, between x, y there may be derived equations of the first, second, third, &c. orders, viz. by equating the prime functions, second derivatives, third derivatives, &c. of all its terms.

Thus, suppose $s = \sin. x$; from this primitive equation we have,

I. Derivative, $s' - \cos. x \,.\, x' = o$,

II. Derivative, $s'' - \cos. x \,.\, x'' + \sin. x \,.\, x'^2 = o$,

which, by substituting for $\cos. x$, and $\sin. x$, their values in the preceding equations, becomes,

$$s''x' - s'x'' + sx'^3 = o.$$

And if we suppose s to be a function of x, since, in that case, $x' = 1$, and $x'' = o$, we shall have this very simple equation,

$$s + s'' = o.$$

The use of derivative equations will be sufficiently apparent from the following examples.

EXAMPLE 1.

To find the sine and versed sine of an arc in terms of the arc.

Put $s =$ sine, $v =$ versed sine; and let each of these

be considered as a function of the arc x; then since $\overline{1 - v}$ = cosine, we have,

$$\left.\begin{aligned} s' &= 1 - v, \\ v' &= s, \end{aligned}\right\} \text{ therefore, } v'' = 1 - v.$$

Assume $v = ax^2 + bx^4 + cx^6 + dx^8 + \&c.$

then $v' = 2ax + 4bx^3 + 6cx^5 + 8dx^7 + \&c.$

$v'' = 2a + 3.4bx^2 + 5.6x^4 + 7.8dx^6 + \&c.$

$= 1 - v = 1 - ax^2 - bx^4 - cx^6 - \&c.$

Whence, by equating the corresponding terms, we deduce these values of the co-efficients a, b, c, &c. viz.

$$a = \tfrac{1}{2},\ b = -\frac{a}{3.4},\ c = -\frac{b}{5.6},\ d = -\frac{c}{7.8},\ \&c.$$

therefore,

$$v = \tfrac{1}{2}x^2 - \frac{1}{2.3.4}x^4 + \frac{1}{2.3.4.5.6}x^6 - \frac{1}{2\ldots\ldots 8}x^8 + \&c.$$

And, because $s = v'$,

$$s = x - \frac{1}{2.3}x^3 + \frac{1}{2.3.4.5}x^5 - \frac{1}{2\ldots 7}x^7 + \&c.$$

This series may also be deduced from the second derivative equation found above, viz. $s + s'' = o$.

N. B.—The series for v must be so taken, that the terms of the series for v'' may have the same powers of x as the corresponding terms of the series for $1 - v$:

thus ax^2 must be the first term in v, to make its second function $2a$ correspond with 1, the first term in $(1-v)$; and as the second term of the latter series is ax^2, the second term of the former must be bx^4, whose second derivative function involves the same power of x, &c.

EXAMPLE 2.

To find an arc in terms of its tangent, and *vice versâ*.

Here $t' = (1+t^2)x'$ (37. 24.); and supposing x a function of t, $x' = 1 - t^2x'$.

Assuming $x = at + bt^3 + ct^5 + dt^7 + et^9 +$ &c.

we have $x' = a + 3bt^2 + 5ct^4 + 7dt^6 + 9et^8 +$ &c.

$= 1 - t^2x' = 1 - at^2 - 3bt^4 - 5ct^6 - 7dt^8 -$ &c.

Whence, $a=1$, $b=-\frac{1}{3}$, $c=+\frac{1}{5}$, $d=-\frac{1}{7}$, $e=+\frac{1}{9}$, &c.

and $x = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 -$ &c.

Again, supposing t a function of x, we have $t' = 1 + t^2$.

Assume $t = ax + bx^3 + cx^5 + dx^7 + ex^9 +$ &c.

then $t' = a + 3bx^2 + 5cx^4 + 7dx^6 + 9ex^8 +$ &c.

$= 1 + t^2 = 1 + a^2x^2 + 2abx^4 + 2ac \left.\begin{matrix} \\ + b^2 \end{matrix}\right\} x^6 + 2ad \left.\begin{matrix} \\ + 2bc \end{matrix}\right\} x^8 + 2ae \left.\begin{matrix} \\ + 2bd \\ + c^2 \end{matrix}\right\} x^{10}$

Whence $a = 1$, $b = \frac{a^2}{3}$, $c = \frac{2ab}{5}$, $d = \frac{2ac + b^2}{7}$,

$e = \frac{2ad + 2bc}{9}$, $f = \frac{2ae + 2bd + c^2}{11}$, $g = \frac{2af + 2be + 2cd}{13}$,

&c. where the law of continuation is manifest.

And $t = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 +$ &c.

EXAMPLE 3.

To find an arc in terms of its sine.

If $s = \sin. x$, then $s' = \sqrt{1 - s^2} \cdot x'$, and $s'^2 + s^2x'^2 - x'^2 = o$; or supposing x a function of s, $1 + s^2x'^2 - x'^2 = o$, of which the second derivative divided by $2x'$ is $sx' + s^2x'' - x'' = o$.

Assume $x = s + as^3 + bs^5 + cs^7 + ds^9 +$ &c.

then $sx' = s + 3as^3 + 5bs^5 + 7cs^7 + 9ds^9 +$ &c.

$+ s^2x'' = +2.3as^3 + 4.5bs^5 + 6.7cs^7 + 8.9ds^9 +$ &c.

$- x'' = -2.3as - 4.5bs^3 - 6.7cs^5 - 8.9ds^7 - 10.11es^9 -$ &c.

$= o.$

Whence $a = \frac{1}{2.3}$, $b = \frac{3^2a}{4.5}$, $c = \frac{5^2b}{6.7}$, $d = \frac{7^2c}{8.9}$, &c.

And,

$$x = s + \frac{1^2}{1.2.3}s^3 + \frac{1^2.3^2}{1.2.3.4.5}s^5 + \frac{1^2.3^2.5^2}{1.....7}s^7 + \frac{(1..7)^2}{1.....9}s^9 + \text{&c.}$$

EXAMPLE 4.

To find an arc in terms of its versed sine.

Let $v = \text{ver. } x$, then $v' = \sqrt{2v - v^2} \cdot x'$, and $v'^2 - 2vx'^2 + v^2x'^2 = o$; or, supposing x a function of v, $1 - 2vx'^2 + v^2x'^2 = o$, of which the next derivative divided by $2x'$ is,

$$x' + 2vx'' - vx' - v^2x'^2 = o.$$

To find what power of v the first term of the series for x must involve, suppose $x = v^n$, and consequently $x' = nv^{n-1}$, $x'' = n \cdot n - 1.v^{n-2}$; let these values be substituted in the above equation, and the whole divided by nv^{n-1}, and we shall have,

$$1 + 2(n-1) - v - (n-1)v = o;$$

which equation, as it must hold for every value of v, must subsist when $v = o$; in which case,

$$1 + 2(n-1) = o, \text{ or } n = \tfrac{1}{2}.$$

Assume then $x = av^{\frac{1}{2}} + bv^{\frac{3}{2}} + cv^{\frac{5}{2}} + dv^{\frac{7}{2}} + \&c.$

Or, to avoid fractional indices, put $y = v^{\frac{1}{2}}$, then

$$v = y^2,\ 1 = 2yy',\ y' = \frac{1}{2y}, \text{ and } y'' = -\frac{1}{4y^3}.$$

$$\left.\begin{array}{rl} x' & \frac{a}{2y} + \frac{3}{2}by + \frac{5}{2}cy^3 + \frac{7}{2}dy^5 + \frac{9}{2}ey^7 + \&c. \\ -vx' & \quad -\frac{1}{2}ay - \frac{3}{2}by^3 - \frac{5}{2}cy^5 - \frac{7}{2}dy^7 - \&c. \\ +2vx'' & -\frac{a}{2y} + \frac{3}{2}by + \frac{15}{2}cy^3 + \frac{35}{2}dy^5 + \frac{63}{2}ey^7 + \&c. \\ -v^2x'' & \quad +\frac{1}{4}ay - \frac{3}{4}by^3 - \frac{15}{4}cy^5 - \frac{35}{4}dy^7 - \&c. \end{array}\right\} = o.$$

Whence $b = \frac{a}{12}$, $c = \frac{3a}{160}$, $d = \frac{5a}{7.128}$, &c.

To find a, which is still undetermined, we have, from the first equation, $x' = \frac{1}{\sqrt{2v - v^2}} = \frac{1}{y\sqrt{2 - y^2}}$.

But, as above, $x' = \frac{a}{2y} + \frac{3}{2}by + \&c.$

therefore $\frac{1}{\sqrt{2 - y^2}} = \frac{a}{2} + \frac{3}{2}by^2 + \&c.$ which, when $y = o$, gives $a = \sqrt{2}$. Hence,

$$x = \sqrt{2v}\left(1 + \frac{v}{3.4} + \frac{3v^2}{5.32} + \frac{5v^3}{7.128} + \&c.\right)$$

EXAMPLE 5.

To find the m^{th} power of the polynome $a + bx + cx^2 + dx^3 + \&c.$

Put $y =$ polynome as before (40. 1.), and $y^m = v$, of which the first derivative equation is $v' - my^{m-1}y' = o$.

Hence $yv' - mvy' = o$.

But $y = a + bx + cx^2 + dx^3 + ex^4 + \&c.$

and $y' = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4 + \&c.$

Assume $v = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$

then $v' = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c.$

which values of y, y', v, v' being substituted in the pre-

ceding equation, and the terms arranged according to the powers of x, we have,

$(aB - mbA) + (2aC + bB - 2mcA - mbB) \cdot x +$
$(3aD + 2bC + cB - 3mdA - 2mcB - mbC)x^2 +$ &c.

Hence the following equations:

$$B = \frac{mbA}{a}.$$

$$C = \frac{2mcA + \overline{m-1} \cdot bB}{2a}.$$

$$D = \frac{3mdA + \overline{2m-1} \cdot cB + \overline{m-2} \cdot bC}{3a}.$$

$$E = \frac{4meA + \overline{3m-1} \cdot dB + \overline{2m-2} \cdot cC + \overline{m-3} \cdot bD}{4a},$$

&c. &c.

from which the co-efficients of the required series may be determined, (the law of continuation being manifest,) provided the first term A can be found. Now, by supposing $x = o$, it is evident that $A = v = y^m = a^m$.

SECTION VII.

TO FIND THE VALUE OF A FRACTION WHOSE TERMS VANISH TOGETHER.

42. ANOTHER use of derivative functions is to determine the value of a fraction, (such as $\frac{x^3-a^3}{x-a}$,) whose terms are functions of x, and, when x becomes equal to some given quantity (a), both vanish together.

Let $\frac{fx}{Fx}$ represent such a fraction, as that, on the supposition of $x = a$, both $fx = o$ and $Fx = o$.

Put $y = \frac{fx}{Fx}$; then $yFx = fx$; which equation, when $x = a$, becomes $o = o$, whatever be the value of y. But its first derivative gives $y'Fx + yF'x = f'x$; and, since $Fx = o$, $yF'x = f'x$. Hence $y = \frac{f'x}{F'x}$. So that in place of the primitives fx, Fx, we may substitute their primes $f'x$, $F'x$; and, by consequence, instead of the latter may be substituted the second derivatives $f''x$, $F''x$, &c.

Thus, $y = \frac{fx}{Fx} = \frac{f'x}{F'x} = \frac{f''x}{F''x} = \frac{f'''x}{F'''x} = \&c.$

From some one of these equivalent fractions the value of y may be ascertained.

If the numerator of one of the fractions vanish, (when x = a given quantity,) and the denominator do not, $y = o$; or if the denominator vanish, and not the numerator, $y = \infty$.

But if one of the terms do not vanish without the other, y must be equal to some determinate quantity; for the derivative functions cannot all successively vanish, by making $x = a$, because then $f(a+i)$ would also vanish, whatever be the value of i, which is impossible. They may, however, become infinite; but, in such cases, the terms of the fraction may be so changed, without altering its value, as to avoid this result.—See Example 9.

EXAMPLE 1.

To find the value of $\frac{x^n - 1}{x - 1}$, when $x = 1$.

Taking the prime functions of both terms, we have $y = \frac{nx^{n-1}}{1}$, which, when $x = 1$, becomes $y = n$.

EXAMPLE 2.

To find $(y =) \frac{x^3 - ax^2 - a^2x + a^3}{x^2 - a^2}$, when $x = a$.

Here $y = \frac{3x^2 - 2ax - a^2}{2x} = \frac{o}{2a}$; therefore $y = o$.

EXAMPLE 3.

To find $(y =) \frac{2x^3 - 5a^2x + 3a^3}{2ax - x^2}$, when $x = a$.

Here $y = \frac{6x^2 - 5a^2}{2a - 2x} = \frac{a^2}{o}$; therefore $y = \infty$.

EXAMPLE 4.

Given $\frac{1 - \sin. x + \cos. x}{\sin. x + \cos. x - 1} = y$, to find y, when $x = 90°$.

$$y = \frac{-\cos. x - \sin. x}{\cos. x - \sin. x} = (\text{when } x = 90°) \frac{-1}{-1} = 1.$$

EXAMPLE 5.

To find $y = \frac{a - x - al.a + al.x}{a - \sqrt{2ax - x^2}}$, when $x = a$.

$$y = \frac{-1 + \frac{a}{x}}{-(2ax - x^2)^{-\frac{1}{2}}.(a - x)} = \frac{(a-x).\sqrt{2ax - x^2}}{-x.(a-x)}$$

$$= -\frac{a}{a} = -1.$$

EXAMPLE 6.

To find $y = \frac{x^x - x}{1 - x + l \,.\, x}$, when $x = 1$.

$$\frac{f'x}{F'x} = \frac{x^x.(1+l.x)-1}{-1+\frac{1}{x}}, \text{ and } \frac{f''x}{F''x} = \frac{x^x(1+l.x)^2 + x^x.\frac{1}{x}}{-\frac{1}{x^2}} = -2.$$

EXAMPLE 7.

To find $y = \frac{a^x - b^x}{x}$, when $x = o$.

$$y = \frac{l \,.\, a \,.\, a^x - l \,.\, b \,.\, b^x}{1} = (\text{when } x = o)\ l \,.\, a - l \,.\, b.$$

EXAMPLE 8.

To find the value of $\frac{x^3 - 4ax^2 + 7a^2x - 2a^3 - 2a^2\sqrt{2ax - a^2}}{x^2 - 2ax - a^2 + 2a\sqrt{2ax - x^2}}$, when $x = a$.

Put $z = \sqrt{2ax - a^2}$, $v = \sqrt{2ax - x^2}$, and the derivatives of both terms are,

$$\frac{3x^2 - 8ax + 7a^2 - 2a^2z'}{2x - 2a + 2av}, \left| \frac{6x - 8a - 2a^2z''}{2 + 2av''}, \right| \frac{3 - a^2z'''}{av'''}, \left| - \frac{az^{IV}}{v^{IV}}, \text{ &c.} \right.$$

Now $z^2 = 2ax - a^2$.

Hence $z' = \frac{a}{z}$, $z'' = -\frac{a^2}{z^3}$, $z''' = \frac{3a^3}{z^5}$, $z^{IV} = -\frac{15a^4}{z^7}$, &c.

$v^2 = 2ax - x^2$, $v' = \frac{a - x}{v} = (\text{putting } \overline{a - x} = u)\ \frac{u}{v}$.

therefore $v'' = -\frac{1}{v} - \frac{u^2}{v^3}$, $v''' = \frac{3u}{v^3} + \frac{3u^3}{v^5}$, $v^{IV} = -\frac{3}{v^3}$, &c.

But, when $x = a$, $z = a$, $v = a$, and $u = o$.

Hence $z' = 1$, $z'' = -\frac{1}{a}$, $z''' = +\frac{3}{a^3}$, $z^{IV} = -\frac{15}{a^3}$, &c.

Also $v' = o$, $v'' = -\frac{1}{a}$, $v''' = o$, $v^{IV} = -\frac{3}{a^3}$, &c.

So that both terms of the fraction vanish in the first, second and third derivative functions; but in the next we have,

$\frac{f^{IV}x}{Fx^{IV}} = -\frac{az^{IV}}{v^{IV}} = +\frac{15a}{a^5} \times -\frac{a^5}{3} = -5a$, the value required.

EXAMPLE 9.

Given $\frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^3 - a^3}}$, to find its value, when $x = a$.

Here the prime function of the numerator =

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x-a}} = \infty .$$

And the prime function of the denominator =

$$\frac{x}{2\sqrt{x-a}} = \infty .$$

On the same supposition of $x = a$, the succeeding derivatives are all infinite.

But, if we multiply both terms of the fraction by $\sqrt{x} + \sqrt{a}$, we get this equivalent fraction, viz.

$$\frac{x-a+\sqrt{x-a}(\sqrt{x}+\sqrt{a})}{\sqrt{x-a}.\sqrt{x+a}(\sqrt{x}+\sqrt{a})}, \text{ or } \frac{\sqrt{x-a}+\sqrt{x}+\sqrt{a}}{\sqrt{x+a}(\sqrt{x}+\sqrt{a})},$$

which, when $x = a$, is evidently $= \frac{1}{\sqrt{2a}}$.

SECTION VIII.

OF MAXIMA AND MINIMA.

43. LET it be required to determine when the function fx can be a maximum or a minimum.

Put $y = fx$, and let i be a very small increment or decrement of x; then, when y is a maximum, it must be greater than either of the functions $f(x+i)$ and $f(x-i)$; and, when a minimum, it must be less than either of them. But,

$$f(x+i) = y + y' \cdot i + y'' \cdot \frac{i^2}{2} + y''' \cdot \frac{i^3}{2.3} + \&c.$$

$$f(x-i) = y - y' \cdot i + y'' \cdot \frac{i^2}{2} - y''' \cdot \frac{i^3}{2.3} + \&c.$$

In which two series, whether y', y'', y''', &c. be positive or negative, the second, fourth, sixth, &c. terms will have unlike signs, and the third, fifth, seventh, &c. like signs. And as i may always be taken so small, that any one of the terms shall be greater than the sum of all the following in the same series, the sign of the leading term only need be regarded, as it will shew whether the sum of the series is to be added or subtracted.

Now, as long as y' has a real value, it is evident that y can neither be greater than both of these functions $f(x+i)$ and $f(x-i)$, nor less than both, being always of an intermediate magnitude between

$$y + y' \cdot i + \&c.$$

$$\text{and } y - y' \cdot i + \&c.$$

But, if $y' = o$, and $y'' < o$, (or negative,) then is y greater than either

$$y - y'' \cdot \frac{i^2}{2} + \&c.$$

$$\text{or, } y - y'' \cdot \frac{i^2}{2} - \&c.$$

Or, if $y' = o$, and $y'' > o$, (or positive,) then is y less than either

$$y + y'' \cdot \frac{i^2}{2} + \&c.$$

$$\text{or, } y + y'' \cdot \frac{i^2}{2} - \&c.$$

that is, y is, in the former case, a maximum; in the latter a minimum.

In like manner, if $y' = o$, and $y'' = o$, there can be no maximum, unless at the same time $y''' = o$, and $y^{IV} < o$; nor minimum, unless $y''' = o$, and $y^{IV} > o$.

To find, then, the value of x, when a given function thereof fx is a maximum or minimum, we have only to find its prime function $f'x$, and, from the equation, $f'x = o$, to determine the value of x. And to know whether the said function fx be a maximum or minimum, (when that circumstance is not rendered evident by the nature of the problem,) we have only to take the next derivative function, and find whether its value (from that of x already determined) be negative or affirmative.

But if the second derivative vanish, there can be no maximum or minimum, unless the third derivative also vanish; in which case we are to proceed with the third and fourth derivatives as with the first and second: And so on.

EXAMPLE 1.

Given the side of a right cone ($= a$), to find its altitude when the content is a maximum.

Let x = alt. required, and c = circumference of a circle, whose diameter is 1; then $a^2 - x^2$ = square of the radius of the base, $ca^2 - cx^2$ = area of the base, and $\frac{1}{3}ca^2x - \frac{1}{3}cx^3$ = solidity of the cone: this function of x being a maximum, we have its prime function,

$$\tfrac{1}{3}ca^2 - cx^2 = o,$$

where $x = \frac{1}{3}a\sqrt{3}$.

Here the second derivative, $-2cx = -\frac{2}{3}ca\sqrt{3}$, is negative, as it must always be in the case of a maximum; but, by the nature of the problem, it is evident that there can be no minimum.

EXAMPLE 2.

To find the value of x when x^x is a minimum.

Let $y = x^x$, then $l.y = xl.x$, and taking their prime functions $\frac{y'}{y} = 1 + l.x$: when therefore $y' = o$, $1 + l.x = o$, or $1 = -l.x = l.\frac{1}{x}$, but $1 = l.e$, therefore $\frac{1}{x} = e$, and

$$x = \frac{1}{e} = \frac{1}{2.718, \&c.} = .3678, \&c.$$

Here the second derivative,

$$y'' = 1 + 2l.x + (l.x)^2 + \frac{x^x}{x} = \frac{x^x}{x},$$

(because $l.x = -1$), is an affirmative quantity, as it

must always be in the case of a minimum ; but in this example it is evident there can be no maximum.

EXAMPLE 3.

To find the value of x, when $x^{\frac{1}{x}}$ is a maximum or minimum.

Put $y = x^{\frac{1}{x}}$, then $l.y = \frac{1}{x} l.x$, and $\frac{y'}{y} = \frac{1}{x^2} - \frac{1}{x^2}.l.x$,

from which, by supposing $y' = o$, we get $l.x = 1$, and $x = e = 2.718$, &c,

Again, taking the second derivatives, we have,

$$\frac{y''}{y} - \frac{y'^2}{y^2} = -\frac{2}{x^2} + \frac{2}{x^2} l.x - \frac{1}{x^2}, \text{ or (since } y' = o, \& \; l.x = 1)$$

$$\frac{y''}{y} = -\frac{1}{x^2} = -\frac{1}{e^2}.$$

y'' then being negative, $x^{\frac{1}{x}}$ must be a maximum, when $x = 2.718$, &c.

EXAMPLE 4.

To divide a given number (a) into two such parts, that the product of the m^{th} power of the one into the n^{th} power of the other shall be a maximum.

We have $x^m(a-x)^n$ a maximum, and therefore $mx^{m-1}(a-x)^n - nx^m(a-x)^{n-1} = o$; whence, $m(a-x) = nx$. So that the given number must be divided in the ratio of m to n, for the parts required.

EXAMPLE 5.

To divide a given number (a) into three such parts, that the product of given powers (m, n, p,) of these parts may be a maximum.

We have $x+y+z=a$, and $x^m y^n z^p$ a maximum. It appears, from the preceding problem, that if z were constant, x and y may be found by dividing $a-z$ in the ratio of m to n. Whatever value, therefore, we give to z, we shall have these values of x and y,

$$x = \frac{m.(a-z)}{m+n}, \quad y = \frac{n.(a-z)}{m+n}.$$

Hence $\frac{m^m(a-z)^m}{(m+n)^m} \cdot \frac{n^n.(a-z)^n}{(m+n)^n} \cdot z^p$ must be a maximum, therefore also $(a-z)^{m+n}.z^p$ a maximum; but this quantity is a maximum (by the preceding problem) when $a-z : z = m+n : p$. Consequently $z = \frac{ap}{s}$, $x = \frac{am}{s}$, and $y = \frac{an}{s}$ (putting $m+n+p=s$).

EXAMPLE 6.

To find the parts into which a given number (a) being divided, their product shall be a maximum.

It follows from the preceding problem, that if a given number is to be divided into two or more parts, so that their product shall be a maximum, the parts must be all equal.

Let x be one of the equal parts, $\frac{a}{x}$ will be their number, and $x^{\frac{a}{x}}$ their continual product.

Put $y = x^{\frac{a}{x}}$, then $l.y = \frac{a}{x}.l.x$, and $\frac{y'}{y} = \frac{a}{x^2} - \frac{a}{x}.l.x$.

But $y' = o$, therefore, $\frac{a}{x^2} - \frac{a}{x^2}.l.x = o$, from which we have $l.x = 1 = l.e$, and $x = 2.718$, &c.

If 17, for example, be the number to be divided, and the parts be required in whole numbers; as 3 is the nearest whole number to 2.718, &c., we shall have $3 \times 3 \times 3 \times 3 \times 3 \times 2 = 486$ for the maximum in this case.

EXAMPLE 7.

To divide a given arc (a) into two such parts, that the product of any given powers (m, n) of their sines may be a maximum.

The two parts being x and $a - x$, the product $\sin.^m x \,.\, \sin.^n (a - x)$ must be a maximum; therefore $m \,.\, l \,.\, \sin. x + n \,.\, l \,.\, \sin. (a - x)$ a maximum.

Hence $\frac{m.\cos.x}{\sin.x} - \frac{n.\cos.(a-x)}{\sin.(a-x)} = o$, and

$m \,.\, \tan (a - x) = n \,.\, \tan. x$, from which we have this analogy, $m + n : m \sim n = \sin. a : \sin.$ difference of the required arcs: Consequently the arcs themselves are given.

SECTION IX.

TO FIND THE LIMITS OF THE DERIVATIVE SERIES CONTINUED INDEFINITELY FROM A GIVEN TERM.

44. Of the infinite series, which composes the developement of $f(x + i)$, let any number of terms from the beginning be denoted by n, and the sum of the remaining terms by R: And let it be required to find the limits of the value of R.

Putting Q for $\frac{i^n}{1 \ldots n} f\left(x + \frac{i}{r}\right)$, we have

$$R = \frac{i^n}{1 \ldots n} f^N x + \frac{i^{n+1}}{1 \ldots n+1} f^{N+1} x +$$

$$\frac{i^{n+2}}{1 \ldots n+2} f^{N+2} x + \frac{i^{n+3}}{1 \ldots n+3} f^{N+3} x + \&c.$$

$$Q = \frac{i^n}{1 \ldots n} f^N x + \frac{i^{n+1}}{1 \ldots n \cdot r} f^{N+1} x +$$

$$\frac{i^{n+2}}{1 \ldots n \cdot r \cdot 2r} f^{N+2} x + \frac{i^{n+3}}{1 \ldots n \cdot r \cdot 2r \cdot 3r} f^{N+3} x + \&c.$$

Now, supposing first the derivative functions to be all affirmative, and comparing the corresponding terms of these series, it is evident that, if $r = n+1$, the first and second terms of the latter must be the same with those of the former, and that every succeeding term of the latter must be less than the corresponding term of the former;

$$\text{therefore, } R > \frac{i^n}{1 \ldots n} f^N\left(x + \frac{i}{n+1}\right).$$

Also, if $r = 1$, the first terms only are equal, and every succeeding term of the latter is greater than the corresponding term of the former;

$$\text{therefore, } R < \frac{i^n}{1 \ldots n} \cdot f^N (x + i).$$

Again, if the derivative functions be alternately affirmative and negative, by taking $f(x-i)$ instead of $f(x+i)$, we may, in some cases at least, have the terms which form the developement of $f(x-i)$ all affirmative or all negative, the primitive function excepted. Hence

R terms of $f(x-i)$ will, in these cases, have similar limits, viz.

$$R > \frac{i^n}{1 \ldots n} f^N\left(x - \frac{i}{n+1}\right), \text{ and } R < \frac{i^n}{1 \ldots n} f^N(x-i),$$

or, *vice versâ*, when the terms are negative.

45. Such being the limits of R terms of $f(x+i)$, when i is of any magnitude whatever, (supposing the derivative functions all affirmative or all negative); the second limit may be brought nearer to the first, if we suppose i to be so small, that any term of the series for R to be greater than the sum of all the following terms. For, in that case, if $r = n + 1$, it is evident that the first term, together with twice the second term of the series for Q, will be greater than R; and consequently $\frac{i^n}{1 \ldots n} f^N\left(x + \frac{2i}{r}\right)$, which is greater than the former, will be greater than the latter. Thus,

$$R > \frac{i^n}{1 \ldots n} f^N\left(x + \frac{i}{n+1}\right), \text{ and } R < \frac{i^n}{1 \ldots n} f^N\left(x + \frac{2i}{n+1}\right).$$

46. The value of R, then, being always of an intermediate magnitude between $\frac{i^n}{1 \ldots n} f^N\left(x + \frac{i}{n+1}\right)$, and

$\frac{i^n}{1 \ldots n} f^N(x+i)$, may be expressed by $\frac{i^n}{1 \ldots n} f^N(x+u)$; where u denotes some quantity greater than $\frac{i}{n+1}$, but less than i. And if the value of u were known for any term, the developement of $f(x+i)$ might then be expressed by a finite series, as thus,

$$f(x+i) = fx + if'x + \frac{i^2}{2} f''x + \frac{i^3}{2.3} f'''(x+u).$$

Its value, however, can only be approximated; and, in general, the nearest approximation is $\frac{i}{n+1}$, when we make the series terminate at the n^{th} derivative.

EXAMPLE.

Required the limits of the value of R, supposing it to begin with the $(n+1)$ term of the developement of the function $(x+a)^m$.

Here the primitive is x^m, and its n^{th} derivative is $m.\overline{m-1}.\overline{m-2}.\ldots\overline{m-n+1}.x^{m-n}$. Hence the $(n+1)$ term of $(x+a)^m$, or first term of R, is

$$\frac{m.\overline{m-1}\ldots\overline{m-n+1}}{1\ldots.n} a^n x^{m-n}.$$

And $R > \frac{m.\overline{m-1}\ldots.\overline{m-n+1}.a^n}{1\ldots.n}\left(x+\frac{a}{n+1}\right)^{m-n}$.

$$R < \frac{m \cdot \overline{m-1} \ldots \ldots \overline{m-n+1} \cdot a^n}{1 \ldots . n} (x+a)^{m-n}.$$

Thus, if R begin with the fifth term of $(x+a)^7$, we have,

$m=7$, $n=4$, and $R = 35x^3a^4 + 21x^2a^5 + 7xa^6 + a^7$.

$R > 35a^4\left(x+\frac{a}{5}\right)^3$, or $35x^3a^4 + 21x^2a^5 + \frac{21}{5}xa^6 + \frac{7}{125}a^7$.

$R < 35a^4(x+a)^3$, or $35x^3a^4 + 105x^2a^5 + 135xa^6 + 35a^7$.

And, if we suppose a very small in respect to x,

$R < 35a^4\left(x+\frac{2a}{5}\right)^3$, or $35x^3a^4 + 42x^2a^5 + \frac{28}{5}xa^6 + \frac{56}{125}a^7$.

In the above example, m is supposed to be a positive integer; for if m were negative, the derivative function would be alternately negative and affirmative: hence the developement of $(x-a)^{-m}$ will have all its terms affirmative. Or, if m be a proper fraction, the derivative functions are alternately affirmative and negative: hence the developement of $(x-a)^m$ will, in that case, have all its terms, after the first, negative.

Thus, for R terms of the function $(x-a)^m$, in which m is either a negative integer or a proper fraction, we have,

$$R > \frac{m \cdot \overline{m-1} \ldots . \overline{m-n+1} \cdot a^n}{1 \ldots . n} \left(x - \frac{a}{n+1}\right)^{m-n}.$$

$$R < \frac{m \cdot \overline{m-1} \ldots \ldots \overline{m-n+1} \cdot a^n}{1 \ldots . n} \cdot (x-a)^{m-n},$$

in the former case: And in the latter,

$$R > \frac{m \cdot \overline{m-1} \ldots \ldots \overline{m-n+1} \cdot a^n}{1 \ldots . n} (x-a)^{m-n}.$$

$$R < \frac{m \cdot \overline{m-1} \ldots \ldots \overline{m-n+1} \cdot a^n}{1 \ldots \ldots n} \left(x - \frac{a}{n+1}\right)^{m-n}.$$

47. Lastly, let it be required to find the limits of the value of R, whatever be the signs of the terms which compose that quantity. Put

$$Q = \frac{i^n}{1 \ldots n} f^N(x+i),$$

and, as before,

$$R = \frac{i^n}{1 \ldots n} f^N x + \frac{i^{n+1}}{1 \ldots n+1} f^{N+1} x + \&c.$$

then, since

$$Q = \frac{i^n}{1 \ldots n} f^N x + \frac{i^{n+1}}{1 \ldots n} f^{N+1} x + \&c.$$

it is evident that the corresponding terms of R and Q will have like signs, whether i be affirmative or negative; for they contain the same power of i, as well as the same derivative function of fx. Now, supposing i so small, that the second terms of R and Q are each greater than the sum of all the following terms in the same series, the common sign of the second terms will

shew whether the first terms are to be increased or diminished. In the former case, it is plain that

$$R > \frac{i^n}{1 \ldots n} f^N x, \text{ and } R < \frac{i^n}{1 \ldots n} f^N(x + i);$$

in the latter, that

$$R < \frac{i^n}{1 \ldots n} f^N x, \text{ and } R > \frac{i^n}{1 \ldots n} f^N(x + i),$$

whether the signs of the leading terms in R and Q be both affirmative or both negative.

48. In all cases, $Q = \frac{i^n}{1 \ldots n} f^N\left(x + \frac{i}{n+1}\right)$ gives a near approximation to the value of R; the first two terms in each being the same, and R being greater or less than Q, according as $i^{n+2} \cdot f^{N+2}x$ is affirmative or negative.

SECTION X.

TO APPROXIMATE THE ROOT OF ANY EQUATION.

49. It appears from the preceding investigation, that when i is very small in respect to the functions of x, the value of $f(x+i)$, whether i be affirmative or negative, may be approximated as follows, (46.) viz.

$f(x+i) = fx + if'(x+\frac{1}{2}i)$ nearly;

$f(x+i) = fx + i.f'x + \frac{i^2}{2} f''(x+\frac{1}{3}i)$ more nearly;

$f(x+i) = fx + i.f'x + \frac{i^2}{2}f''x + \frac{i^3}{2.3} f'''(x+\frac{1}{4}i)$ still more nearly, &c. &c.

The same will easily appear from the developement of the final functions, each of which is equivalent to more than two of the remaining terms of the series.

Now, let $x+v$ represent the true root of any equation, and x such an approximation to the same, that v may be small in respect of x.

$$\text{Suppose } f(x+v) = \mathrm{M},$$
$$\text{and } fx = \mathrm{N}.$$

Then $v.f'x+\frac{v^2}{2}f''x+\frac{v^3}{2.3}f'''x+\&c. = \mathrm{D}\,(=\mathrm{M}-\mathrm{N})$.

Hence we have the following approximations to the value of v, denoted by v', v'', v''', &c.

$$v' = \frac{\mathrm{D}}{f'x}, \text{ nearly};$$

$$v'' = \frac{\mathrm{D}}{f'x+\frac{v'}{2}f''x}, \text{ more nearly};$$

$$v''' = \frac{\mathrm{D}}{f'x+\frac{v''}{2}f''(x+\frac{1}{3}v')}, \text{ more nearly, \&c.}$$

The last value of v, according as it proves affirmative or negative, being added to, or subtracted from x, gives a very near approximation to the true root, provided x itself be assumed within less than an unit of the same.

Again, since $v^2f''x + 2vf'x = 2\mathrm{D}$ nearly,
and $v^2f''(x+\frac{1}{3}v) + 2vf'x = 2\mathrm{D}$ more nearly,
we have $v' = \dfrac{\sqrt{2b\mathrm{D}+a^2}-a}{b}$,

$$v'' = \frac{\sqrt{2c\mathrm{D}+a^2}-a}{c},$$

where $a = f'x$, $b = f''x$, and $c = f''(x+\frac{1}{3}v')$.

Here, if $2bD$ be negative, put $\frac{\sqrt{2bD}}{a} = \text{sin. A},$

$$\text{then } v' = \frac{a}{b} \text{ sin. A. tan. } \tfrac{1}{2}\text{A};$$

if $2bD$ be affirmative, put $\frac{\sqrt{2bD}}{a} = \text{tan. A},$

$$\text{then } v'' = \frac{a}{b} \text{ tan. A. tan. } \tfrac{1}{2}\text{A}.$$

And v''' may be found in the same manner.

EXAMPLE 1.

To find z from the equation,

$$z^4 - 4z^3 + 6z^2 - 6z + 2 = o.$$

By putting $z = 2$, we find the result -2; therefore the true root is greater than 2; but if $z = 3$, the result is $+11$. Thus,

	$z^4 - 4z^3 + 6z^2 - 6z + 2$	$-4 + 6 - 6 + 2$
Sub.	$-2 \quad +4 \quad -4 \quad +4$	$-3 + 3 - 9 - 9$
Rem.	$-2 \quad +2 \quad -2 \quad -2$	$-1 + 3 + 3 + 11$

where each remainder, multiplied by $-z$, (that is, by -2 or -3,) gives the next following subtrahend.

The true root, then, is between 2 and 3, but nearer to the former than to the latter. Assuming, there-

fore, $x = 2.4$, as an approximation to the value of z, $(= x + v,)$ we have,

$$\begin{array}{r} -4 \quad +6 \quad -6 \quad +2 \\ -2.4 \quad +3.84 \quad -5.184 \quad +1.9584 \\ \hline -1.6 \quad +2.16 \quad -0.816 \quad +0.0416 \end{array}$$ for the result.

Hence $f(x + v) = 0$,

$$\begin{aligned} fx &= 0.0416, \\ \mathrm{D} &= -0.0416, \\ f'x &= 4x^3 - 12x^2 + 12x - 6 = 8.976, \\ f''x &= 12x^2 - 24x + 12 = 12(x-1)^2 = 23.52, \end{aligned}$$

$$v' = \frac{\mathrm{D}}{f'x} = \frac{-0.0416}{8.976} = -0.0046,$$

$$v'' = \frac{\mathrm{D}}{f'x + \frac{v'}{2} f''x} = \frac{-0.0416}{8.922} = -0.004663,$$

$$\begin{array}{rl} x = & 2.4 \\ v'' = - & .004663 \\ \hline z = & 2.395337 \end{array}$$

Again, $x + \frac{1}{3}v'' = 2.398446$, $f''(x + \frac{1}{3}v'') = 23.467815$,

And $v''' = \dfrac{\mathrm{D}}{f'x + \frac{v''}{2} f'(x + \frac{1}{3}v'')} = -0.004663007.$

Consequently $z = 2.395336993$, &c.

Otherwise,

The given equation may be easily reduced to this,

$$(z-1)^4-(2z-1)=0.$$

Hence $4\,l.(z-1)-l.(2z-1)=0.$

And, putting $x=2.4$,

$$fx=4l.1.4-l.3.8=\quad 0.010888,$$

$$\mathrm{D}\quad =-.010888,$$

$$f'x=\frac{4}{x-1},-\frac{2}{2x-1}=\frac{6.20}{2.66},$$

$$v'=\frac{\mathrm{D}}{f'x}=\frac{-0.010888\times 2.66}{6.2}=-0.00467,$$

&c. &c. &c.

Thus, by using logarithms, we obtain a nearer approximation to the value of v.

Otherwise,

$2\mathrm{D}=0.0832$	log. 8.9201233		log. 8.9201233
$b=23.52$	log. 1.3714373	c log.	1.3704724
	Sum 0.2915606		2)0.2905957
	½ Sum 0.1457803		0.1452979
$a=8.976$	ar. co. 9.0469172		9.0469172
$\mathrm{A}=8°\ 57'\ 56''$	sin. 9.1926975	$8°\,57'\,21''.12$	sin. 9.1922151
$\frac{1}{2}\mathrm{A}=4°\ 28'\ 58''$	tan. 8.8943120	4 28 40.56	tan. 8.8938407
$\log.\frac{a}{b}$	9.5816455	$\log.\frac{a}{c}$	9.5826104
$v'=-0.004663$	log. 7.6686550	v''	log. 7.6686652

Hence $c=12\,(1.398446)^2$, $v''=-0.004663008$, as before.

EXAMPLE 2.

Given the equation, $z^3+z^2-166z+660=0$, to find z.

Put $x=9$, then $f(x+v) = 0$

$$fx = -24$$

$$D = 24$$

$$f'x = 95 = a.$$

$$f''x = 56 = b.$$

$$v' = \frac{\sqrt{2bD+a^2}-a}{b} = \frac{\sqrt{11713}-95}{56} = 0.2361,$$

$$c = f'(x+\tfrac{1}{3}'v) = 56.4722,$$

$$v'' = \frac{\sqrt{11735.6656}-95}{56.4722} = 0.236068.$$

Hence $z = 9.236068$, &c.

Or thus,

$2D = 48$	log. 1.6812412	log. 1.6812412
$b = 56$	log. 1.7481880	c log. 1.7518348
	2)3.4294292	2)3.4330760
	1.7147146	1.7165380
$a=95$	ar. co. 8.0222764	ar. co. 8.0222764
$A=28°\ 37'\ 24''$	tan. 9.7369910	$28°\ 43'28''.64$ tan. 9.7388144
$\frac{1}{2}A=14\ 18\ 42$	tan. 9.4067337	$14\ 21\ 44.32$ tan. 9.4083338
	$\frac{a}{b}$ log. 0.2295356	$\frac{a}{c}$ log. 0.2258888
$v' = 0.2361$	log. 9.3732603	v'' log. 9.3730370

Hence $c = 56.4722$ $v'' = 0.236068$, as above.

EXAMPLE 3.

Given the equation, $(y^2+1)^{\sqrt{2}} + y = 16$.

Here it is plain that the value of y is between 2 and 3: Assume, then, $x = 2$.

$$\sqrt{2}\,.\,l\,.\,(y^2+1) - l\,.\,(16-y) = 0,$$

$$fx = \sqrt{2}\,.\,l\,.\,5 - l\,.\,14 \quad = -0.3639,$$

$$\mathrm{D} \quad = 0.3639,$$

$$f'x = \frac{2\sqrt{2}.x}{x^2+1} + \frac{1}{16-x} = 1.20,$$

$$v' = \frac{\mathrm{D}}{f'x} = \frac{0.364}{1.2} = 0.3,$$

$$f''x = \frac{2\sqrt{2}}{x^2+1} - \frac{2\sqrt{2}}{x^2+1}\cdot\frac{2x^2}{x^2+1} + \frac{1}{(16-x)^2} = -0.3,$$

$$v'' = \frac{\mathrm{D}}{f'x + \frac{v'}{2} f''x} = \frac{0.3639}{1.153} = 0.315.$$

Hence $x' = 2.315$, with which, the operation repeated, gives,

$$\mathrm{D} = 0.00013746045,\ f'x = 1.102728,$$

$$f''x = -0.2978282, \text{ and } v'' = 0.000124636;$$

therefore $y = 2.315124636$, &c.

EXAMPLE 4.

Given the equation $z + c \sin. z = m$, to find the arc z, or, $z + 0.205513 \sin. z = 60°$.

Substituting z for sin. z, we get $x = \frac{m}{1+c} = \frac{60'}{1.2}$ $= 50°$ for the first approximation to the value of z;

then $z + .205513$ sin. $z = 60°$,

$fx = 50° + .205513$ sin. $50° = 59°\ 1'\ 12''.7$,

$\mathrm{D} = 3527''.3$,

$f'x = 1 + c$ cos. x,

$$v' = \frac{\mathrm{D}}{f'x} = 3115''.3,$$

$$v'' = \frac{\mathrm{D}}{f'(x + \frac{1}{2}v')} = 3119''.0 = 51'\ 59''.$$

Hence $z = 50°\ 51'\ 59''$.

Or thus by logarithms,

Const. log. 5.3144251				
c log. 9.3128388	9.3128			9.3128
50° sin. 9.8842540	cos. 9.8081	cos. $\overline{x+\frac{1}{2}v'}$		9.8041
32472''.7 log. 4.5115179	.1321 log. 9.1209		.1309	9.1169
36000 .0 = 10°	1.1321 ar. co. 9.9461		1.1309	9.9466
3527 .3 = D	log. 3.5474			3.5474
	v' = 3115''.3 log. 3.4935	v'' = 3119.0		3.4940

EXAMPLE 5.

To find that arc which is equal to its cosine.

The arc, it is evident, is something less than 45°. Thus,

$$z - \cos. z = 0,$$
$$fx = 45° - \cos. 45 = 4°.486,$$
$$D = -4.486,$$
$$f'x = 1 + \sin. x = 1.707,$$
$$v' = \frac{D}{f'x} = -\frac{4.486}{1.707} = -2.62.$$

Hence $x' = 42°.38$.

Again, $z - \cos. z = 0,$
$42°.38 - \cos. 42°.38 = 0.056156,$
$f'x = 1 + \sin. x = 1.67404,$
$$v'' = \frac{D}{1.67404} = -0.033545.$$

Hence $z = 42°.346455 = 42° 20' 47''.24$.

EXAMPLE 6.

To find the arc which is the double of its sine.

It is plain that $114° > 2 \sin. 114$, (because $57° = 1$ nearly,) and that $90 < 2 \sin. 90$.

Assuming, then, $x = 102°$, a mean between these, for the first approximation, we have

$$z - 2 \sin. z = 0,$$
$$102° - 2 \sin. 102 = -10°.08,$$
$$D = 10°.08,$$

$f'x = 1 + 2 \cos. x$, (because cos. x is negative).

$$r' = \frac{D}{1 + 2 \cos. x} = 7'.12, \text{ and}$$

$$r' = \frac{D}{1 + 2 \cos. (x + \frac{1}{2}r)} = 6'.56.$$

Hence $x = 108'.56$.

Again, $z - 2 \sin. z = 0$,

$108'.56 - 2 \sin. 108.56 = -0.0717$,

$$r' = \frac{D}{1 + 2 \cos. x} = 0.0488.$$

Hence $z = 108'.6088 = 108° 36' 13''.7$.

EXAMPLE 7.

Required the fifth root of the number 161900, to ten or more places of decimals, by a common table of logarithms.

$f'x = 5x^4$.

Log. 5.2092468 $\quad z^5 = 161900.$

1.0418494 $\quad \overline{11.01157}^5 = 161899.765466873823749$

$D = 0.234533126762\overline{51}$

x^4 log. 4.1673974 $\quad r' = 0.00000,319034$

$\frac{D}{5}$ log. 8.6712342 $\quad x = 11.01157$

$r' = \frac{D}{5x^4}$ log. 4.5038368 $\quad x' = 11.01157,319034$

As we cannot have, from the tables, a more exact value of the fraction $\frac{D}{5x^4}$, if more places of decimals are required, the operation must be repeated, using x', instead of x, to get the next value of D. But, to save the trouble of raising x', (which contains thirteen figures,) to the fifth power, we have taken D to fourteen places of decimals, in order to find D′, in the following manner.

Because $D' = z^5 - x'^5 = z^5 - (x + v')^5 = D - 5x^4v' - 10x^3v'^2$, (the other terms being too small to be regarded,) we have only to multiply x^4 by $5v$.

The product is,	0.23453,29947,317
to which, adding $10x^3v'^2$, (found by logs.)	1355,901
and subtracting the amount,	0.23453,31303,218
from D =	0.23453,31267,625
we get D′ =	— 0.00000,00035,593

Then $\frac{D'}{f'x'} = \frac{D'}{5x'^4}$, found by logarithms, gives,

$v'' =$	— 0.00000,00000,00048,417
But $x' =$	11.01157,31903,40000,000
Therefore $z =$	11.01157,31903,39951,583, &c.

EXAMPLE 8.

From the equation $x^x = 100$, to find x.

Here, as the true value of x is between 3 and 4, let 3 be the first approximation,

$$x \,.\, l \,.\, x = l \,.\, 100 = 4.60,$$
$$3 \,.\, l \,.\, 3 = l \,.\, 27 = 3.30,$$
$$\mathrm{D} = 1.30,$$
$$f'x = 1 + l \,.\, x = 2.10,$$
$$v' = \frac{1.30}{2.10} = 0.6.$$

Hence $x = 3.6$ for the second approximation, with which, repeating the operation, we have

$$x\, l \,.\, x = 4.605170186,$$
$$3.6\, l \,.\, 3.6 = 4.611361844,$$
$$\mathrm{D} = -.006191658,$$
$$f'x = 1 + l \,.\, x = 2.2809338,$$
$$f''x = \frac{1}{x} = \frac{10}{36}.$$

$$v' = -\frac{.0061916}{2.2809} = -0.00271,$$

$$\frac{v'}{2} f''x = -0.000376,$$

$$v'' = -\frac{.006191658}{2.2805574} = -0.002715,$$

$$\frac{v''}{2} f''(x + \tfrac{1}{3}v'') = -0.0003772,$$

$$v''' = -\frac{.006191658}{2.2805566} = -0.00271497, \text{ and}$$

$$x = 3.59728503, \text{ \&c.}$$

The same method of approximation may be extended to compound equations. Thus, if

$$\begin{aligned} f(x+v) + \mathrm{F}(y+z) &= \mathrm{M}, \\ fx \quad\quad + \mathrm{F}y \quad\quad &= \mathrm{N}, \text{ and} \\ \varphi(\overline{x+v}, y+z) \quad\quad &= \mathrm{P}, \\ \varphi(x, y) \quad\quad &= \mathrm{Q}, \end{aligned}$$

then $vf'x + z\mathrm{F}'y = \mathrm{D}$, and $v \,.\, \varphi'(x) + z\varphi'(y) = \Delta$ nearly, when v and z are very small. Hence we have two equations of this form,

$$\begin{aligned} av + bz &= \mathrm{D}, \\ cv + dz &= \Delta; \end{aligned}$$

whence we get

$$v = \frac{d\mathrm{D} - b\Delta}{ad - bc},$$

$$z = \frac{a\Delta - c\mathrm{D}}{ad - bc}.$$

EXAMPLE 9.

Given the equations $x^2 + y^2 = 300$, and $x^4 + y^4 = 150$, to find x and y.

Having found, by trial, that 4 and 3 are approximate values of x and y in these equations, we have

$$\left.\begin{array}{l} x^4 + y^3 = 300 \\ 4^4 + 3^3 = 283 \end{array}\right\} \text{ and } \left\{\begin{array}{l} x^3 + y^4 = 150 \\ 4^3 + 3^4 = 145 \end{array}\right.$$

$$\overline{ax + bz = 17} \qquad \overline{cv + dz = 5}$$

Now, a = prime function of $x^4 = \overline{1 + l.x}.x^4 = 610.8$,

b = prime function of $y^3 = \overline{1 + l.y}.y^3 = 56.7$,

c = prime function of $x^3 + y^4$, when y is constant, $= x^3 . \frac{y}{x} + y^4 . l . y = 137.0$,

d = prime function of $x^3 + y^4$, when x is constant, $= y^4 . \frac{x}{y} + x^3 . l . x = 196.7$;

therefore $6108\,v + 567z = 170$,

$1370\,v + 1967z = 50$.

Hence $v = 0.0272$, $z = 0.0064$, $x' = 4.027$, $y' = 3.006$.

Again, with these values of x and y, we find,

$D = -0.4041$, $\Delta = +0.03442$; $a = 653.44$, $b = 57.435$, $c = 141.738$, $d = 204.416$; $v = -0.000674$, and $z = +0.000636$;

therefore $x = 4.026326$, $y = 3.006636$, &c.

SECTION XI.

OF TANGENTS AND CONTACTS.

50. A LINE of any kind is said to be a tangent to a curve, when it meets the curve at one point, in such a manner, that no other line of the same kind can be drawn between them from the point of concourse; which point is therefore named the point of contact.

Let $y = fx$ be the equation of any curve whatever, x being the absciss, and y the ordinate; also, $z = Fx$ the equation of a line touching the former, and referred to the same axis. As they have one point common, y must be equal to z, and consequently $fx = Fx$. Now, to compare the courses of the line and curve after meeting, let i be any increment of the common absciss x; then to the absciss $\overline{x + i}$, $f(x + i)$ and $F(x + i)$ are the corresponding ordinates; the difference of which, viz. $f(x + i) - F(x + i)$, is the distance of two points in the said line and curve, corresponding to the

same absciss. But that difference, since $fx - Fx = o$, may be resolved into

$$i(f'x - F'x) + \frac{i^2}{2}(f''x - F''x) + \frac{i^3}{2.3}(f'''x - F'''x) + \text{\&c.}$$

from which it is evident, that, if the prime functions are equal, (or, $f'x - F'x = o$,) the distance of the points will be less, and therefore the line and curve will more nearly approach one another; that, if the second derivatives are also equal, (or, $f''x - F''x = o$,) the distance will be still less, and the approach of the line and curve towards a coincidence still nearer: And so on.

But, in order to examine more particularly into these different orders of contact, (or approaches towards a coincidence,) let $v = \varphi x$ be the equation of a third line referred to the same axis with the former; and let one point be common to all; that is, let the ordinates y, z and v coincide; then $fx = Fx = \varphi x$.

Let D be the difference of the ordinates of the first two lines to the absciss $(x + i)$; and Δ the difference of the first and third to the same absciss; that is,

$$\text{let } D = f(x + i) - F(x + i),$$
$$\text{and } \Delta = f(x + i) - \varphi(x + i).$$

It is evident that the third line cannot pass between

the first two, unless D, for any value of i, however small, be greater than Δ : But, since

$$D = i(f'x - F'x) + \frac{i^2}{2}(f''(x+u) - F''(x+\hat{u})), \text{ and}$$

$$\Delta = i(f'x - \varphi'x) + \frac{i^2}{2}(f''(x+u) - \varphi''(x+\bar{u}));$$

supposing the first two lines to be such, that $f'x = F'x$, the value of D becomes,

$$D = \frac{i^2}{2}(f''(x+u) - F''(x+\hat{u}));$$

whence it is easy to see, that so long as the multiple of i, in the expression for Δ, is a positive quantity, we may always take i so small, that $f'x - \varphi'x$ shall be greater than $\frac{i}{2}(f''(x+u) - F''(x+\hat{u}))$, and consequently Δ greater than D. Thus the third line cannot pass between the first two, (which have $f'x = F'x$,) unless it have $\varphi'x = f'x$.

In like manner, when the first two lines have $f'x = F'x$, and $f''x = F''x$, the third cannot pass between them, unless it have, at the same time, $\varphi'x = f'x$, and $\varphi''x = f''x$.

In general, therefore, if any two curves whatever, having their axis in one straight line; and the same

variable abscis x have also two corresponding ordinates, y, z, coincident, (so that the curves have one point common); and if the prime functions of y, z be equal; or, their primes equal, and their second functions also equal; or, their prime, second and third functions respectively equal, &c.; then it is impossible for any third curve, drawn through the common point, to pass between them, if it have not the same derivative functions of its ordinate v equal to the corresponding ones of the ordinates y, z.

This theory we shall now apply to the most simple cases of contact, viz. when the tangent is a straight line, or a circle.

1. STRAIGHT LINES.

51. Let the given equation of the line which forms the contact be $y = Fx = b + \frac{bx}{a}$, the equation of a straight line. This line forms a contact of the first order with the curve whose equation is $y = fx$.

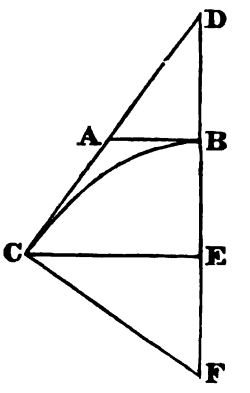
If $a : b = 1 : t$, then $\frac{b}{a} = t$, the tangent of a given angle. And the two constant quantities b, t, (which

may be termed the elements of contact,) are determined by these two equations:

$$\text{Primitive,} \quad y - b - tx = o.$$
$$\text{Derivative,} \quad y' - t = o.$$

Whence
$$t = y',$$
$$b = y - xy'.$$

Let BC be the curve whose equation is $y = fx$; BE, EC two co-ordinates, of which the absciss BE $= x$, and the ordinate EC $= y$: Also, let BD $= a$, AB $= b$, be co-ordinates of the straight line AC, which has one ordinate CE in common with the curve BC. Then $y = b + tx$ is the equation of the straight line AC; and, in order that AC may be a tangent to the curve at C, the prime functions of their ordinates must be equal, from which the above relations between the constant and variable quantities are derived.



It is plain that $y' = t$ is the tangent of the angle D, which the straight line AC makes with the axis of the curve; and therefore $\frac{y}{y'}$ = DE, the subtangent. Hence also the subnormal EF, (being a third proportional to

the subtangent DE and the ordinate EC,) is expressed by yy'.

Or, considering x and y as functions of some third variable, that is, substituting $\frac{y'}{x'}$ in place of y', we have these general rules for determining the tangent to any curve whose equation is $y = fx$:

$$\frac{yx'}{y'} = \text{subtangent},$$

$$\frac{yy'}{x'} = \text{subnormal};$$

which we shall now illustrate by a few examples.

EXAMPLE 1.

To draw a tangent from a given point in the curve of a parabola.

The equation being $y^2 = px$, we have $2yy' = px'$. Hence the subnormal $\frac{yy'}{x'} = \frac{1}{2}p = \frac{1}{2}$ the parameter of the axis; and the subtangent $= \frac{yx'}{y'} = \frac{2y^2}{p} = 2x =$ double the absciss; from either of which the tangent may be drawn.

EXAMPLE 2.

To draw a tangent to the ellipse or hyperbola.

Let a = semi-transverse, and b = semi-conjugate axis; the equations of the two curves will be,

$$y^2 = \frac{b^2}{a^2}(2ax \mp x^2)\text{; from which we have,}$$

$$2yy' = \frac{b^2}{a^2}(2ax' \mp 2xx').$$

Hence, the subnormal $\frac{yy'}{x'} = \frac{b^2}{a^2}(a \mp x)$,

and subtangent $\frac{yx'}{y'} = \frac{2ax \mp x^2}{a \mp x}$.

If $a = b$ $(= r)$, the ellipse becomes a circle, whose equation is $y^2 = 2rx - x^2$; and the subnormal being $r - x$, the normal, (or perpendicular to the tangent at the point of contact,) is therefore a radius of the circle; which is the well-known property of its tangent.

EXAMPLE 3.

Let the given curve be the cissoid of Diocles, of which the equation is $y^2 = \frac{x^3}{a-x}$.

Here $2yy' = \frac{\overline{a-x}.3x^2x' + x^3x'}{(a-x)^2}$, and $\frac{yx'}{y'} = \frac{2y^2.(a-x)^2}{\overline{a-x}.3x^2+x^3} = \frac{2x^3(a-x)}{(a-x)3x^2+x^3} = \frac{2x(a-x)}{3a-2x}$ = subtangent.

EXAMPLE 4.

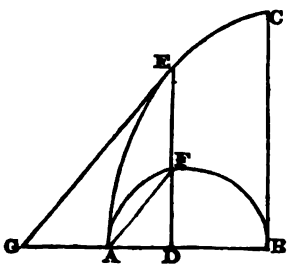
Let the equation of the curve be $\frac{c}{a}.y^{m+n}=(a-x)^m.x^n$.

The subtangent is $\frac{(m+n).(ax-x^2)}{(a-x)}$.

EXAMPLE 5.

To draw a tangent from a given point in the cycloid.

Let AB be the axis of the cycloid ABC, and E a given point in the curve from which it is required to draw a tangent.



Draw the ordinate DE intersecting the semicircle upon AB in F: put AB $= 2r$, AD $= x$, DE $= y$, and the arc AF $= z$: Also, let s, c, v, denote the sine, cosine, and supplemental versed sine of the arc z to the radius r.

By the property of the cycloid, we have $y = s + z$, therefore $y' = s' + z' = \frac{cz'}{r} + z'$ (37. 25.) $= \frac{vz'}{r}$. But $sz' = rx'$ (37. 25). Hence $y' = \frac{vx'}{s}$, and the subtangent $\frac{yx'}{y'} = \frac{sy'}{v}$, that is, supposing the tangent EG to

meet the axis AB in G, $GD = \frac{DE.DF}{BD}$: consequently $DE . DG = BD : DF = DF : DA$, and the tangent EG parallel to the chord AF.

II. CIRCLES.

52. Next, let the circumference of a circle be the line which forms the contact. Its equation may be expressed thus,

$$(x - a)^2 + (y - b)^2 - r^2 = o,$$

where r denotes the radius, and x, y the variable parts of two rectangular co-ordinates, $(x - a)$, $(y - b)$.

This line is susceptible of a contact of the second order, the elements thereof $(a, b, r,)$ being determined by these three equations:

Primitive, $(x-a)^2+(y-b)^2-r^2=o.$

I. Derivative, $(x-a).x'+(y-b).y'=o.$

II. Derivative, $x'^2+(x-a)x''+y'^2+(y-b).y''=o.$

From the first two we have

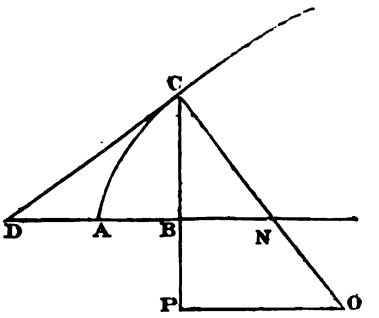
$$r^2 - (x - a)^2 = (y - b^2) = \frac{(x-a)^2.x'^2}{y'^2}.$$

Hence, $\pm ry' = (x-a)\sqrt{x'^2+y'^2}$.

In like manner, $\pm rx' = (y-b)\sqrt{x'^2+y'^2}$.

By substituting these values of $(x-a)$, $(y-b)$, in the last we find $r = \pm \frac{(x'^2+y'^2)^{\frac{1}{2}}}{(x'y''+y'x'')} = \pm \frac{(1+y'^2)^{\frac{1}{2}}}{y''}$, (if $x'=1$), whence a and b may be also determined.

The first two equations serve to determine only a contact of the first order (as in the preceding case), which may be formed with the curve AC at any point C by a circle of any radius: for the second equation determines the locus of the centres of such circles. Thus, CD being the straight line which touches the curve at C, and CNO perpendicular to CD, meeting AB in N, let PO be drawn parallel to AB to meet the ordinate CB in P, and the normal CN in O; then, from the second equation we have $b-y : a-x = 1 : y' =$ rad. : tan. D = CP : PO; if therefore CP $= b-y$, then PO $= a-x$, and CNO the locus of the centres of all the tangent circles. But, of all these circles, that which forms the most intimate contact is determined by the third equation: And its radius is named the radius of curvature, because the



curve has, at the point of contact, the same curvature with the circle.

A more simple expression for (r) the radius of curvature, may be obtained thus :

Since the subnormal BN $= \frac{yy'}{x'}$, the normal

$$\text{CN} = \sqrt{y^2 + \frac{y^2y'^2}{x'^2}} = \frac{y}{x'}\sqrt{x'^2+y'^2} = n, \text{ (suppose) ;}$$

$$\text{hence } r = \pm\frac{(nx')^3}{y^2(x'y''+y'x'')} = \pm\frac{n^3}{y^2y''}, \text{ (if } x' = 1).$$

EXAMPLE 1.

To find the radius of curvature at a given point in the curve of a parabola.

From the equation of the curve $y^2 = px$, we have $2yy' = p$, $y' = \frac{p}{2y}$, and $y'' = -\frac{p^2}{4y^3}$. Hence,

$$r = -\frac{n^3}{y^2y''} = \frac{4n^3}{p^2}.$$

Thus $\text{CO} = \frac{4\text{CN}^2}{4\text{NB}^2} = \frac{\text{DN.CN}}{\text{NB}}$, whence CO : DN = CN : NB = CO : OP, and OP = DN = half the parameter of the diameter at the point of contact, from which the centre O of the required circle may be determined.

EXAMPLE 2.

To find the radius of curvature at a given point of an ellipse or hyperbola.

From the equation $y^2 = \frac{b^2}{a^2}(2ax \mp x^2)$ we have

$$yy' = \frac{b^2}{a^2}(a \mp x) = s \text{ the subnormal.}$$

Hence $y' = \frac{s}{y}$, and $y'' = \frac{s'}{y} - \frac{sy'}{y^2}$; but $s' = \mp\frac{b^2}{a^2}$;

therefore $y^3y'' = \mp\frac{b^2}{a^2} \cdot y^2 - s^2 = -\frac{b^4}{a^2}$.

Consequently $r = -\frac{n^3}{y^3y''} = \frac{a^2n^3}{b^4}$.

At the vertex of the axis, where x and y vanish together, s and n coincide, and become each equal to $\frac{b^2}{a}$, half the parameter of the axis, which therefore is the radius of curvature at that point, as appears by substituting $\frac{b^2}{a}$ for n in the general expression of r.

At the vertex of the conjugate axis of the ellipse, s the subnormal vanishes, and the normal $n = y = b$ the semi-conjugate. Hence $r = \frac{a^2b^3}{b^4} = \frac{a^2}{b}$, half the parameter of the conjugate.

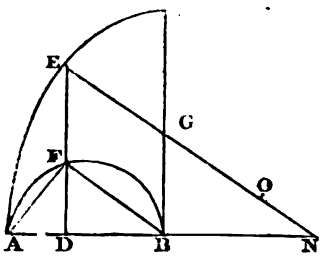
Lastly, in all the three conic sections, the radius of curvature at any point is to the normal at the same

point, in the duplicate ratio of the said normal to half the parameter of the axis.

EXAMPLE 3.

To find the radius of curvature at a given point of a cycloid.

Let AD $= x$, DE $= y$, AF $= z$, DF $= s$, DB $= v$, and c = cos. z to the radius $r = \frac{1}{2}$AB: then if y, z, &c. be functions of x, it appears from Example 5, p. 82, that $y' = \frac{v}{s}$; but $vx = s^2$, hence



$y' = \frac{s}{x}$, and $y'' = \frac{s'}{x} - \frac{s}{x^2}$; but $s' = \frac{cx'}{r} = \frac{c}{s}$, therefore $-y'' = \frac{s}{x^2} - \frac{c}{sx} = \frac{s^2 - cx}{sx^2} = \frac{vx - cx}{sx^2} = \frac{r}{sx}$. Consequently EO $= -\frac{n^3}{y^3 y''} = \frac{sx}{r} \cdot \frac{\text{EN}^3}{\text{DE}^3}$: But $\frac{\text{EN}^3}{\text{DE}^3} = \frac{\text{FB}^3}{\text{DF}^3}$, for the tangent at E being parallel to the chord AF, the normal EN must be parallel to the chord FB.

Thus, EO $= \frac{sx.\text{FB}^3}{r.s^3}$: But FB$^2 = 2rv$, and $s^2 = vx$, therefore EO, the radius of curvature at E, is double the chord FB or EG.

SECTION XII.

DEVELOPEMENT OF THE FUNCTION OF A POLYNOME.

53. In the first three sections, it has been shewn how the function of a binome, as $f(x + i)$, may be developed into a series of this form, $A + Bi + Ci^2 + Di^3 +$ &c.; in which A represents the primitive function fx, and B, C, D, &c. its derivatives, with certain divisors: Also, how the derivatives may be obtained, each from the preceding function after a similar manner, in the several kinds of analytical functions: And, in the fourth section, in what cases the derivatives become infinite.

In the fifth and following sections, the method of derivation has been applied to several useful purposes, and exemplified in the solution of various problems.

We return to the theory of derivation; and propose now to point out a general method of developing any function of a polynome, as,

$$F(x + \xi i + \gamma i^2 + \delta i^3 + \varepsilon i^4 + \&c.)$$

of a definite or an indefinite number of terms, into a series of the same form as before, viz.

$$A + Bi + Ci^2 + Di^3 + \&c.$$

54. If $fx = y$, then $f(x+i) = y + y' \cdot i + \frac{1}{2}y'' \cdot i^2 + \frac{1}{2.3}y''' \cdot i^3 +$ &c.; therefore $F(y + y' \cdot i + \frac{1}{2}y'' \cdot i^2 + \frac{1}{2.3}y'' \cdot i^3 +$ &c.$) = Ff(x+i) =$ (putting φ for Ff) $= \varphi x + \varphi' x \cdot i + \frac{1}{2}\varphi'' x \cdot i^2 + \frac{1}{2.3}\varphi''' \cdot i^3 +$ &c.

Here $\varphi x = Ffx = Fy$; $\varphi' x = F'y \cdot y'$; $\varphi'' x = F'y \cdot y'' + F''y \cdot y'^2$; $\varphi''' x = F'y \cdot y''' + 3F''y \cdot y'' \cdot y' + F'''y' \cdot y'^3$, &c.

Thus any function of this polynome, $y + y' \cdot i + \frac{1}{2}y'' \cdot i^2 +$ denoted by F, may be developed into

$$Fy + (Fy)' \cdot i + \tfrac{1}{2}(Fy)'' \cdot i^2 + \tfrac{1}{2.3}(Fy)''' + \text{ &c.};$$

or, expanding the derivatives, into

$$Fy + F'y \cdot y' \cdot i + \tfrac{1}{2}(F'y \cdot y'' + F'''y \cdot y'^2) i^2 + \tfrac{1}{2.3}(F'y \cdot y''' + 3F''y \cdot y'' \cdot y' + F'''y \cdot y'^3) i^3 + \text{ &c.}$$

55. Now the developement of any function of a polynome, whose terms are independent, is perfectly similar, $F(x + \xi i + \gamma i^2 +) = Fx + (Fx)' \cdot i + \frac{1}{2}(Fx)'' \cdot i^2 + \frac{1}{2.3}(Fx)''' \cdot i^3 +$ &c. provided we expand the derivatives $(Fx)'$, $(Fx)''$, &c. as if x were itself the function of some other variable, and for x' we substitute ξ; for x'', 2γ; for x''', 2.3δ; for x^{iv}, $2.3.4\varepsilon$, &c. &c. For

$$F(x+\mathfrak{G}i)=Fx+F'x.\mathfrak{G}.i+\tfrac{1}{2}F''x.\mathfrak{G}^2.i^2+\frac{1}{2.3}F'''x.\mathfrak{G}^3.i^3+\&c.;$$

in which substituting $\mathfrak{G}+\gamma i$ in place of $\mathfrak{G}$, we get

$$F(x+\mathfrak{G}i+\gamma i^2) = Fx+F'x.\mathfrak{G}i+\tfrac{1}{2}(F'x.2\gamma+F''x.\mathfrak{G}^2)\,i^2+$$
$$\frac{1}{2.3}(3F''x.2\mathfrak{G}\gamma+F'''x.\mathfrak{G}^3)\,i^3+\&c.;$$

in which again substituting $\gamma+\delta i$ for γ, we have

$$Fx+\mathfrak{G}i+\gamma i^2+\delta i^3) = Fx+F'x.\mathfrak{G}i+\tfrac{1}{2}(F'x.2\gamma+F''x.\mathfrak{G}^2)\,i^2$$
$$+\frac{1}{2.3}F'x.2.3\delta+3F''x.2\mathfrak{G}\gamma+F'''x.\mathfrak{G}^3)\,i^3+\&c.$$

56. Thus it appears, that the developement of the two functions,

$$\varphi(\alpha+z)$$
$$\varphi(\alpha+\mathfrak{G}z+\gamma z^2+\delta z^3+\&c.$$

where α is considered as a function of x, only differ in this, that in expanding the derivative functions of $\varphi\alpha$ in the latter, we must substitute $\mathfrak{G}$ for α'; 2γ for α''; 2.3δ for α'''; &c.

In order, therefore, to distinguish these, (which may be termed Polynomial Derivatives,) from common derivative functions, we shall represent them thus:

$\left|\frac{\varphi\alpha}{1}\right|'$, $\left|\frac{\varphi\alpha}{1}\right|''$, $\left|\frac{\varphi\alpha}{1}\right|'''$, &c., or, with their proper divisors thus, $\left|\frac{\varphi\alpha}{1}\right|'$, $\left|\frac{\varphi\alpha}{1.2}\right|''$, $\left|\frac{\varphi\alpha}{1.2.3}\right|'''$, or, since the index shows

the number of factors in the divisor, the polynomial derivatives with their divisors may be more concisely expressed thus : $\left|\frac{\varphi\alpha}{\cdot}\right|', \left|\frac{\varphi\alpha}{\cdot}\right|'', \left|\frac{\varphi\alpha}{\cdot}\right|''', \left|\frac{\varphi\alpha}{\cdot}\right|^{N}$, &c.

Here, however, it must be carefully observed, that as the first derivative of $\left|\frac{\varphi\alpha}{1.2.3}\right|'''$ is $\left|\frac{\varphi\alpha}{1.2.3}\right|^{IV}$, so the first derivative of $\left|\frac{\varphi\alpha}{\cdot}\right|'''$ is not $\left|\frac{\varphi\alpha}{\cdot}\right|^{IV}$, but $4\left|\frac{\varphi\alpha}{\cdot}\right|^{IV}$, its second derivative $4.5\left|\frac{\varphi\alpha}{\cdot}\right|^{V}$, &c.

Hence the following theorem :

57. $\varphi(\alpha + \mathfrak{C}z + \gamma z^2 + \delta z^3 +)$

$= \varphi\alpha + \left|\frac{\varphi\alpha}{\cdot}\right|' . z + \left|\frac{\varphi\alpha}{\cdot}\right|'' . z^2 + \left|\frac{\varphi\alpha}{\cdot}\right|''' . z^3 +$ &c.

$= A + B . z + C . z^2 + D . z^3 +$ &c.

where $A = \varphi\alpha$, $B = A' = (\varphi\alpha)'$, $2C = B' = A'' = (\varphi\alpha)''$, $2.3D = 2C' = B'' = A''' = (\varphi\alpha)'''$, $2.3.4E = (\varphi\alpha)^{IV}$,

or $4E = D$, &c.

And in the developement of $(\varphi\alpha)'$, $(\varphi\alpha)''$, &c. $\mathfrak{C} = \alpha'$, $2\gamma = \mathfrak{C}' = \alpha''$, $2.3\delta = 2\gamma' = \mathfrak{C}'' = \alpha'''$, or $3\delta = \gamma'$, &c. &c.

So that this method of derivation requires, that each co-efficient of a term of the polynome, $\alpha + \mathfrak{C}z + \gamma z^2 +$, &c. or of the series $A + Bz + Cz^2 +$, &c. when multiplied by the exponent of z in that term, should be taken

for the prime function of the preceding co-efficient; that is, $4E - D'$, $5F = E'$, &c. $4\varepsilon = \delta'$, $5\xi = \varepsilon'$, &c.

EXAMPLE 1.

To develope the function, log. $(\alpha+\zeta z+\gamma z^2+\delta z^3+\varepsilon z^4+)$ &c. into a series of this form, $A+Bz+Cz^2+Dz^3+$ &c.

Here $A = l.\alpha$; and since $(l.\alpha)' = \frac{\alpha'}{\alpha}$, $B = A' = \frac{\zeta}{\alpha}$; therefore $\alpha B = \zeta$. Then, from this primitive equation, we have, according to the method of derivation explained above, these successive derivatives:

1. $\alpha B' + B\alpha' = \zeta'$, that is, $2C\alpha + B\zeta = 2\gamma$, or $C\alpha + \frac{1}{2}B\zeta = \gamma$.
2. $C\alpha' + \alpha C' + \frac{1}{2}B\zeta' + \frac{1}{2}\zeta B' = \gamma'$,
   or, $C\zeta + 3D\alpha + B\gamma + C\zeta = 3\delta$,
   or, $D\alpha + \frac{2}{3}C\zeta + \frac{1}{3}B\gamma = \delta$. And in like manner,
3. $E\alpha + \frac{3}{4}D\zeta + \frac{2}{4}C\gamma + \frac{1}{4}B\delta = \varepsilon$.
4. $F\alpha + \frac{4}{5}E\zeta + \frac{3}{5}D\gamma + \frac{2}{5}C\delta + \frac{1}{5}B\varepsilon = \xi$. &c.

The law of continuation is manifest: And the value of each of the indeterminate quantities B, C, D, &c. is easily deduced from the values of those which precede. Thus we find the given function,

$$= l.\alpha + \frac{\zeta}{\alpha}.z + \frac{2\alpha\gamma - \zeta^2}{2\alpha^2}.z^2 + \frac{3\alpha^2\delta - 3\alpha\zeta\gamma + \zeta^3}{3\alpha^3}.z^3 +$$
$$\frac{4\alpha^3\varepsilon - 4\alpha^2\zeta\delta - 2\alpha^2\gamma^2 + 4\alpha\zeta^2\gamma - \zeta^4}{4\alpha^4}.z^4 + \text{ &c.}$$

Cor. 1. Hence, if $\alpha = \epsilon = \gamma = \delta =$ &c., we shall have $l.(\alpha + \alpha z + \alpha z^2 + \alpha z^3 +$ &c.)

$$= l.\alpha + z + \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 + \tfrac{1}{4}z^4 + \text{&c.}$$

Thus, $l.\frac{10}{3}$, or $l.(3 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} +$ &c.)

$$= l.3 + \tfrac{1}{10} + \tfrac{1}{200} + \tfrac{1}{3000} + \tfrac{1}{40000} + \text{&c.};$$

whence $2l.3 = l.10 - \frac{1}{10} - \frac{1}{200} - \frac{1}{3000} - \frac{1}{40000} -$ &c.

Cor. 2. Hence also, if $\epsilon = \gamma = \delta =$ &c. putting $\alpha - \epsilon = d$.

$$l.(\alpha + \epsilon z + \epsilon z^2 + \epsilon z^3 + \text{&c.})$$

$$= l.\alpha + \frac{\epsilon}{\alpha}.z + \frac{\alpha^2 - d^2}{2\alpha^2}.z^2 + \frac{\alpha^2 - d^2}{3\alpha^3}.z^3 + \frac{\alpha^4 - d^4}{4\alpha^4}.z^4 + \text{&c.}$$

Thus $l.\frac{16}{3}$, or, $l.(5 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} +$ &c.)

$$= l.5 + \frac{3}{5.10} + \frac{21}{2.5^2.10^2} + \frac{117}{3.5^3.10^3} + \text{&c.}$$

whence, $5l.2 = l.10 + l.3 + \frac{3}{5.10} + \frac{21}{2.5^2.10^2}$

$$+ \frac{117}{3.5^3.10^3} + \frac{609}{4.5^4.10^4} + \text{&c.}$$

Cor. 3. Again, if γ, ε, $\varkappa$, &c. vanish, we shall have $B = \frac{\epsilon}{\alpha}$, $C = -\frac{1}{2}\frac{\epsilon^2}{\alpha^2}$, $D = \frac{\delta}{\alpha} + \frac{1}{3}.\frac{\epsilon^3}{\alpha^3}$, $E = -\left(\frac{\epsilon\delta}{\alpha^3} + \frac{1}{4}.\frac{\epsilon^4}{\alpha^4}\right)$, $F = \frac{\xi}{\alpha} + \frac{\epsilon^2\delta}{\alpha^3} + \frac{1}{5}.\frac{\epsilon^5}{\alpha^5}$. &c.: And if α, ϵ, δ, ξ, &c. be all equal, $B = 1$, $C = -\frac{1}{2}$, $D = \frac{1}{3}$, $E = -\frac{1}{4}$, $F = \frac{1}{5}$,

$G = -\frac{1}{6}^6$, &c. Thus, log. $\frac{10}{11}^9 = l.9 \cdot 909090$, &c.
$= l.9 + \frac{1}{10} - \frac{1}{200} + \frac{4}{3000} - \frac{6}{40000} +$ &c.

EXAMPLE 2.

Given $e^{\alpha+\zeta z+\gamma z^2+\delta z^3+}$ &c. $= A+B.z+Cz^2+Dz^3+$ &c. to find the co-efficients of the latter series.

Here the hyperbolic logarithm is given in one series, to find the number in another of a similar form. This problem, therefore, is the inverse of the preceding, and is to be resolved by nearly the same process. For, since $e^\alpha = A$, by taking their prime functions, we have $e^\alpha\zeta = B$, or $B = A\zeta$. Hence,

$$C = A\gamma + \tfrac{1}{2}B\zeta.$$
$$D = A\delta + \tfrac{2}{3}B\gamma + \tfrac{1}{3}C\zeta.$$
$$E = A\varepsilon + \tfrac{3}{4}B\delta + \tfrac{2}{4}C\gamma + \tfrac{1}{4}D\zeta.$$
$$F = A\xi + \tfrac{4}{5}B\varepsilon + \tfrac{3}{5}C\delta + \tfrac{2}{5}D\gamma + \tfrac{1}{5}E\zeta.$$
&c. &c.

Thus,

$$e^{\alpha+\zeta z+\text{&c.}} = e^\alpha\left(1+\zeta z+\overline{\gamma+\tfrac{1}{2}\zeta^2}.z^2+\overline{\delta+\zeta\gamma+\tfrac{1}{2.3}\zeta^3}.z^3+\right.$$
$$\overline{\varepsilon+\zeta\delta+\tfrac{1}{2}\gamma^2+\tfrac{1}{2}\zeta^2\gamma+\tfrac{1}{2.3.4}\zeta^4}.z^4+$$
$$\left.\overline{\xi+\zeta\varepsilon+\gamma\delta+\tfrac{1}{2}\zeta^2\delta+\tfrac{1}{2}\zeta\gamma^2+\tfrac{1}{6}\zeta^3\gamma+\tfrac{1}{2.3.5}\zeta^5}.z^5+\text{&c.}\right)$$

If, for example, the number answering to the hyperbolic logarithm, 0.69314, be required; we have $\alpha = o$, (and therefore the general multiplier $e^\alpha = 1$,)

$\mathfrak{C} = 6$, $\gamma = 9$, $\delta = 3$, $\varepsilon = 1$, $\xi = 4$, and $z = \frac{1}{10}$: then

$$A + Bz + Cz^2 + Dz^3 + \&c.$$
$$= 1 + 0.6 + 0.27 + 0.093 + 0.02755 + \&c.$$

EXAMPLE 3.

Given, $\sin.(\alpha + \mathfrak{C}z + \gamma z^2 + \delta z^3 + \&c.)$
$= A + Bz + Cz^2 + Dz^4 + \&c.$

Here $A = \sin.\alpha.$

$B = \cos.\alpha.\mathfrak{C}.$

$C = \cos.\alpha.\gamma - \frac{1}{2}\sin.\alpha.\mathfrak{C}^2.$

$D = \cos.\alpha.\delta - \frac{1}{2}\sin.\alpha.\mathfrak{C}\gamma - \frac{1}{2.3}\cos.\alpha.\mathfrak{C}^3.$

$E = \cos.\alpha.\varepsilon - \frac{1}{2}\sin.\alpha.(2\mathfrak{C}\delta + \gamma^2) - \frac{1}{2.3}\cos.\alpha.3\mathfrak{C}^2\gamma$
$+ \frac{1}{2.3.4}\sin.\alpha.\mathfrak{C}^4.$ &c. &c.

Or, in order to express C, D, E, &c. in terms of the preceding co-efficients, let A be substituted for $\sin.\alpha$, and $\frac{B}{\mathfrak{C}}$ for $\cos.\alpha$, in their respective equations : then,

$C\mathfrak{C} = B\gamma - \frac{1}{2}A\mathfrak{C}^2.$

$D\mathfrak{C} = B\delta - \frac{1}{2}A\mathfrak{C}^2\gamma - \frac{1}{2.3}B\mathfrak{C}^3.$

$E\mathfrak{C} = B\varepsilon - \frac{1}{2}A(2\mathfrak{C}\delta + \gamma^2) - \frac{1}{2.3}B.3\mathfrak{C}^2\gamma + \frac{1}{2.3.4}A.\mathfrak{C}^4.$

&c. &c.

EXAMPLE 4.

Given $(\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c.)^m$
$= A + Bz + Cz^2 + Dz^3 + \&c.$

Here $A = \alpha^m$. $B = m\alpha^{m-1}\beta$; therefore,

$\alpha B = mA\beta$. Hence $2\alpha C + B\beta = 2m\gamma A + mB\beta$, and

$2\alpha C = \overline{m-1}.B\beta + 2mA\gamma$. In like manner,

$3\alpha D = \overline{m-2}.C\beta + 2\overline{m-1}.B\gamma + 3mA\delta$.

$4\alpha E = \overline{m-3}.D\beta + 2\overline{m-2}.C\gamma + 3\overline{m-1}.B\delta + 4mA\varepsilon$.

$5\alpha F = \overline{m-4}.E\beta + 2\overline{m-3}.D\gamma + 3\overline{m-2}.C\delta$
$+ 4\overline{m-1}B\varepsilon + 5m\alpha\xi$. &c. &c.

SECTION XIII.

DEVELOPEMENT OF A POLYNOMIAL DERIVATIVE.

HAVING shewn that any function of a given polynome, as $\varphi(\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c.)$ may be developed into a series of polynomial derivatives, expressed by

$$\left|\frac{\varphi\alpha}{1}\right|^0 . z^0 + \left|\frac{\varphi\alpha}{.}\right|' . z' + \left|\frac{\varphi\alpha}{.}\right|'' . z^2 + \left|\frac{\varphi\alpha}{.}\right|''' z^3 + \text{- - -} \left|\frac{\varphi\alpha}{.}\right|^N . z^n + \&c.$$

and illustrated the method of derivation, by applying

the general theorem to particular examples, in which we have derived each co-efficient of the required series, $Az^0 + Bz + Cz^2 + Dz^3 + Ez^4 + \&c. = A_n . z^n + \&c.$ successively from the preceding (the first co-efficient A being always given $= \varphi\alpha$); we are next to inquire how any proposed term of the developement, as $A_n . z^n$, or $\left|\frac{\varphi\alpha}{.}\right|^N . z^n$, may be expanded independently of the rest.

The knowledge of this will greatly facilitate the method of derivation, which we have endeavoured to explain, by enabling us to reduce to order very complicated expressions.

The general developement of the polynomial derivative, $\left|\frac{\varphi\alpha}{.}\right|^N$, (which is equivalent to the co-efficient A_n), is given by the following theorem:

$$58.\quad \left|\frac{\varphi\alpha}{.}\right|^N = \varphi'\alpha . \left|\frac{\mathfrak{C}}{.}\right|^{N-1} + \tfrac{1}{2}\varphi''\alpha . \left|\frac{\mathfrak{C}^2}{.}\right|^{N-2} + \frac{1}{2.3}\varphi'''\alpha \left|\frac{\mathfrak{C}^3}{.}\right|^{N-3} + \&c. + \frac{1}{1 \ldots n}\varphi^N\alpha . \mathfrak{C}^n.$$

For putting $\pi z = \mathfrak{C}z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \&c.$ we have

$$\pi = \mathfrak{C} + \gamma z + \delta z^2 + \varepsilon z^3 + \&c. \text{ and } (57.)$$

$$\pi^n = \mathfrak{C}^n + \left|\frac{\mathfrak{C}^n}{.}\right|' . z + \left|\frac{\mathfrak{C}^n}{.}\right|'' . z^2 + \left|\frac{\mathfrak{C}^n}{.}\right|''' . z^3 + \&c. \quad \text{But}$$

$$\varphi(\alpha+\pi z)=\varphi\alpha+\varphi'\alpha.\pi z+\tfrac{1}{2}\varphi''\alpha.\pi^2z^2+\frac{1}{2.3}\varphi'''\alpha.\pi^3z^3+\&c.$$

therefore, substituting for the different functions of π, their values as above, we get $\varphi(\alpha+\pi z)=\varphi\alpha+$

$$\left.\begin{array}{l}\varphi'\alpha\mathfrak{C}.z+\varphi'\alpha\left|\frac{\mathfrak{C}}{.}\right|'\\ \quad+\tfrac{1}{2}\varphi''\alpha\mathfrak{C}^2\end{array}\right|z^2+\left.\begin{array}{l}\varphi'\alpha\left|\frac{\mathfrak{C}}{.}\right|''\\ \quad+\tfrac{1}{2}\varphi''\alpha\left|\frac{\mathfrak{C}^2}{.}\right|'\\ \quad+\frac{1}{2.3}\varphi'''\alpha\mathfrak{C}^3\end{array}\right|z^3+\left.\begin{array}{l}\varphi'\alpha\left|\frac{\mathfrak{C}}{.}\right|'''\\ \quad+\tfrac{1}{2}\varphi''\alpha\left|\frac{\mathfrak{C}^2}{.}\right|''\\ \quad+\frac{1}{2.3}\varphi'''\alpha\left|\frac{\mathfrak{C}^3}{.}\right|'\\ \quad+\frac{1}{2.3.4}\varphi^{\text{IV}}\alpha\mathfrak{C}^4\end{array}\right|z^4+\text{ &c.}$$

Thus we have,

$$\left|\frac{\varphi\alpha}{.}\right|'=\varphi'\alpha.\mathfrak{C}.$$

$$\left|\frac{\varphi\alpha}{.}\right|''=\varphi'\alpha\left|\frac{\mathfrak{C}}{.}\right|'+\tfrac{1}{2}\varphi''\alpha.\mathfrak{C}^2.$$

$$\left|\frac{\varphi\alpha}{.}\right|'''=\varphi'\alpha\left|\frac{\mathfrak{C}}{.}\right|''+\tfrac{1}{2}\varphi''\alpha\left|\frac{\mathfrak{C}^2}{.}\right|'+\frac{1}{2.3}\varphi'''\alpha\mathfrak{C}^3.$$

$$\left|\frac{\varphi\alpha}{.}\right|^{\text{IV}}=\varphi'\alpha\left|\frac{\mathfrak{C}}{.}\right|'''+\tfrac{1}{2}\varphi''\alpha\left|\frac{\mathfrak{C}^2}{.}\right|''+\frac{1}{2.3}\varphi'''\alpha\left|\frac{\mathfrak{C}^3}{.}\right|'+\frac{1}{2.3.4}\varphi^{\text{IV}}\alpha\mathfrak{C}^4.$$

$$\left|\frac{\varphi\alpha}{.}\right|^{\text{V}}=\varphi'\alpha\left|\frac{\mathfrak{C}}{.}\right|^{\text{IV}}+\tfrac{1}{2}\varphi''\alpha\left|\frac{\mathfrak{C}^2}{.}\right|'''+\frac{1}{2.3}\varphi'''\alpha\left|\frac{\mathfrak{C}^3}{.}\right|''$$
$$+\frac{1}{2.3.4}\varphi^{\text{IV}}\alpha\left|\frac{\mathfrak{C}^4}{.}\right|'+\frac{1}{2..5}\varphi^{\text{V}}\alpha\mathfrak{C}^5.\quad\&c.\quad\&c.$$

from which the proposition is manifest.

Hence also,

59. $\left|\frac{\varphi\alpha}{.}\right|^{N} = \frac{1}{1\ldots n}\varphi^{N}\alpha\mathfrak{C}^{n} + \frac{1}{1\ldots n-1}\varphi^{N-1}\alpha\left|\frac{\mathfrak{C}^{n-1}}{.}\right|'$

$+\frac{1}{1\ldots n-2}\varphi^{N-2}\alpha\left|\frac{\mathfrak{C}^{n-2}}{.}\right|'' - - - - - + \varphi'\alpha\left|\frac{\mathfrak{C}}{.}\right|^{N-1}$, being the same expression reversed.

Now, by applying the theorem (58.) to particular examples, it will easily appear how the general polynomial derivative, $\left|\frac{\mathfrak{C}^{n}}{.}\right|^{M}$, (in which $\mathfrak{C}$ must be considered as the leading term of a polynome,) may be developed.

60. TABLE I.

1. $\left|\frac{\mathfrak{C}}{.}\right|' = 1.\gamma = \gamma.$

2. $\left|\frac{\mathfrak{C}}{.}\right|'' = 1.\left|\frac{\gamma}{.}\right|' = \delta.$ (by Example 1.)

3. $\left|\frac{\mathfrak{C}}{.}\right|''' = 1.\left|\frac{\gamma}{.}\right|'' = \varepsilon.$ —— (2.)

4. $\left|\frac{\mathfrak{C}}{.}\right|^{IV} = 1.\left|\frac{\gamma}{.}\right|''' = \zeta.$ —— (3.)

5. $\left|\frac{\mathfrak{C}}{.}\right|^{V} = 1.\left|\frac{\gamma}{.}\right|^{IV} = \eta.$ —— (4.)

&c. &c.

6. $\left|\frac{\mathfrak{C}^{2}}{.}\right|' = 2\mathfrak{C}.\gamma = 2\mathfrak{C}\gamma.$

7. $\left|\frac{\epsilon^2}{\cdot}\right|'' = 2\epsilon\left|\frac{\gamma}{\cdot}\right|' + 1.\left|\frac{\gamma^2}{1}\right|^0. = 2\epsilon\delta + \gamma^2.$

8. $\left|\frac{\epsilon^2}{\cdot}\right|''' = 2\epsilon\left|\frac{\gamma}{\cdot}\right|'' + 1.\left|\frac{\gamma^2}{\cdot}\right|'. = 2\epsilon_3 + 2\gamma\delta.$

9. $\left|\frac{\epsilon^2}{\cdot}\right|^{IV} = 2\epsilon\left|\frac{\gamma}{\cdot}\right|''' + 1.\left|\frac{\gamma^2}{\cdot}\right|''. = 2\epsilon\zeta + 2\gamma\epsilon + \delta^2.$

10. $\left|\frac{\epsilon^2}{\cdot}\right|^{V} = 2\epsilon\left|\frac{\gamma}{\cdot}\right|^{IV} + 1.\left|\frac{\gamma^2}{\cdot}\right|'''. = 2\epsilon\eta + 2\gamma\zeta + 2\delta\epsilon.$

&c. &c.

11. $\left|\frac{\epsilon^3}{\cdot}\right|' = 3\epsilon^2.\gamma \qquad = 3\epsilon^2\gamma.$

12. $\left|\frac{\epsilon^3}{\cdot}\right|'' = 3\epsilon^2\left|\frac{\gamma}{\cdot}\right|' + \frac{1}{2}.6\epsilon\left|\frac{\gamma^2}{\cdot}\right|^0. \qquad = 3\epsilon^2\delta + 3\epsilon\gamma^2.$

13. $\left|\frac{\epsilon^3}{\cdot}\right|''' = 3\epsilon^2\left|\frac{\gamma}{\cdot}\right|'' + \frac{1}{2}.6\epsilon\left|\frac{\gamma^2}{\cdot}\right|' + \frac{1}{2.3}.6\left|\frac{\gamma^3}{\cdot}\right|^0. = 3\epsilon^2\epsilon + 6\epsilon\gamma\delta + \gamma^3.$

14. $\left|\frac{\epsilon^3}{\cdot}\right|^{IV} = 3\epsilon^2\left|\frac{\gamma}{\cdot}\right|''' + \frac{1}{2}.6\epsilon\left|\frac{\gamma^2}{\cdot}\right|'' + \frac{1}{2.3}.6\left|\frac{\gamma^3}{\cdot}\right|'. = 3\epsilon^2\zeta + 6\epsilon\gamma\epsilon + 3\epsilon\delta^2 + 3\gamma^2\delta.$

15. $\left|\frac{\epsilon^3}{\cdot}\right|^{V} = 3\epsilon^2\left|\frac{\gamma}{\cdot}\right|^{IV} + \frac{1}{2}.6\epsilon\left|\frac{\gamma^2}{\cdot}\right|''' + \frac{1}{2.3}.6\left|\frac{\gamma^3}{\cdot}\right|'' = 3\epsilon^2\eta + 3\epsilon(2\gamma\zeta + 2\delta\epsilon) + 1.(3\gamma^2\epsilon + 3\gamma\delta^2).$

&c.

16. $\left|\frac{\epsilon^4}{\cdot}\right|' = 4\epsilon^3\gamma.$

17. $\left|\frac{\epsilon^4}{\cdot}\right|'' = 4\epsilon^3\delta + 6\epsilon^2\gamma^2.$

18. $\left|\frac{\mathcal{C}^4}{\cdot}\right|''' = 4\mathcal{C}^3\varepsilon + 6\mathcal{C}^2.2\gamma\delta + 4\mathcal{C}\gamma^3.$

19. $\left|\frac{\mathcal{C}^4}{\cdot}\right|^{IV} = 4\mathcal{C}^3\zeta + 6\mathcal{C}^2(2\gamma\varepsilon + \delta^2) + 4\mathcal{C}.3\gamma^2\delta + \gamma^4.$

20. $\left|\frac{\mathcal{C}^4}{\cdot}\right|^{V} = 4\mathcal{C}\eta + 6\mathcal{C}^2(2\gamma\zeta + 2\delta\varepsilon) + 4\mathcal{C}(3\gamma^2\varepsilon + 3\gamma\delta^2) + 4\gamma^3\delta.$

&c. &c.

In general,

61. $\left|\frac{\mathcal{C}^n}{\cdot}\right|^M = n\mathcal{C}^{n-1}\left|\frac{\gamma}{\cdot}\right|^{M-1} + \frac{n.n-1}{2}\mathcal{C}^{n-2}\left|\frac{\gamma^2}{\cdot}\right|^{M-2} + \frac{n.n-1.n-2}{2.3}\mathcal{C}^{n-3}\left|\frac{\gamma^3}{\cdot}\right|^{M-3} + \&c.$

It appears from the first part of the table, that the polynomial derivative, $\left|\frac{\mathcal{C}}{\cdot}\right|^V = \eta$, that is, = fifth letter after $\mathcal{C}$; therefore the first term in the developement of $\left|\frac{\mathcal{C}^n}{\cdot}\right|^M = n\mathcal{C}^{n-1}$ into the m^{th} letter after $\mathcal{C}$, (or $\overline{m-1}$ letter after γ). The succeeding terms may be resolved according to the examples in the table.

But a more expeditious method of obtaining the value of a polynomial derivative is what we are next to point out.

62. Let it be required to find the co-efficient (G) of the seventh term (Gz^6) in the developement of $\varphi(\alpha' + \epsilon z + \gamma z^2 + \delta z^3 + \&c.)$

By the preceding theorem and table (58. 60.) we find $G = \left|\frac{\varphi\alpha}{\cdot}\right|^{VI} = \varphi'\alpha \cdot \eta + \frac{1}{2}\varphi''\alpha\,(2\epsilon\zeta + 2\gamma\varepsilon + \delta^2) +$

$\frac{1}{2.3}\varphi'''\alpha\,(3\epsilon^2\varepsilon + 6\epsilon\gamma\delta + \gamma^3) + \frac{1}{2.3.4}\varphi^{IV}\alpha(4\epsilon^2\delta + 6\epsilon^2\gamma^2) +$

$\frac{1}{2..5}\varphi^V\alpha \cdot 5\epsilon^4\gamma + \frac{1}{2\ldots6}\varphi^{VI}\alpha\epsilon^6.$

Now, without attending to the derivative functions of $\varphi\alpha$ and their divisors, which are easily obtained, let us take the other factors in this and the next two co-efficients (H, I), and consider how those of one co-efficient may be deduced from those of the preceding.

G...η, $(2\epsilon\zeta + 2\gamma\varepsilon + \delta^2)$, $(3\epsilon^2\varepsilon + 6\epsilon\gamma\delta + \gamma^3)$, $(4\epsilon^2\delta + 6\epsilon^2\gamma^2)$, $5\epsilon^4\gamma$, ϵ^6.

H...θ, $(2\epsilon\eta + 2\gamma\zeta + 2\delta\varepsilon)$, $(3\epsilon^2\zeta + 6\epsilon\gamma\varepsilon + 3\epsilon\delta^2 + 3\gamma^2\delta)$, $(4\epsilon^3\varepsilon + 12\epsilon^2\gamma\delta + 4\epsilon\gamma^3)$, $(5\epsilon^4\delta + 10\epsilon^3\gamma^2)$, $6\epsilon^5\gamma$, ϵ^7.

I...ι, $(2\epsilon\theta + 2\gamma\eta + 2\delta\zeta + \varepsilon^2)$, $(3\epsilon^2\eta + 6\epsilon\gamma\zeta + 6\epsilon\delta\varepsilon + 3\gamma^2\varepsilon + 3\gamma\delta^2)$, $(4\epsilon^3\zeta + 12\epsilon^2\gamma\varepsilon + 6\epsilon^2\delta^2 + 12\epsilon\gamma^2\delta + \gamma^4)$, $(5\epsilon^4\varepsilon + 20\epsilon^3\gamma\delta + 10\epsilon^2\gamma^3)$, $(6\epsilon^5\delta + 15\epsilon^4\gamma^2)$, $7\epsilon^6\gamma$, ϵ^8.

Observe, 1. That the number of terms in the developement of Gz^6, Hz^7, Iz^8, &c. is expressed by the in-

dex of z: And that, of these terms, the first and the last two are simple, the rest compound.

2. That in each simple term, or component part of a term, consisting of two or more letters, the letters are to be arranged according to their order in the alphabet: Thus, though $\zeta^2\gamma\delta^3$, $\gamma\zeta^2\delta^3$, $\delta^3\zeta^2\gamma$, &c. express the same quantity, yet it is only the first of these arrangements which we adopt for the sake of the following rule.

RULE.

63. To derive the terms of one co-efficient (H) from those of the preceding co-efficient (G), the last letter of each quantity is to be changed into the next letter of the alphabet: Thus, in place of η write θ; for $2\zeta\zeta$ write $2\zeta\eta$; for $2\gamma\varepsilon$ write $2\gamma\zeta$; &c. But if the last letter have an index, the quantity must also be multiplied by that index: Thus, for δ^2 $(=\delta\delta)$, write $2\delta\varepsilon$; for γ^3 $(=\gamma^2\gamma)$ write $3\gamma^2\delta$; for $6\zeta^2\gamma^2$, write $12\zeta^2\gamma\delta$; &c.

Further, if the last two letters of any quantity be such as follow one another in the order of the alphabet, an additional quantity must be formed, by changing the first of these two letters into the last, multiplying by the index of the former, and dividing by the in-

creased index of the latter : Thus, $6\zeta^2\gamma^2$ (or $6\zeta\zeta\gamma\gamma$), becomes $12\zeta^2\gamma\delta + 4\zeta\gamma^3$: Here $\zeta\gamma$ being adjacent letters of the alphabet, an additional quantity is necessary, to deduce it from $6\zeta^2\gamma^2$: change ζ into γ, and you have $6\zeta\gamma^3$, which multiplied by 2, the index of ζ, and divided by 3, the increased index of γ, gives $4\zeta\gamma^3$: Thus, also, $2\gamma\delta$ becomes $2\gamma\varepsilon + \delta^2$; $6\zeta\gamma\delta$ becomes $6\zeta\gamma\varepsilon + 3\zeta\delta^2$; &c.

To the terms thus derived must be added one more, viz. the same power of ζ as of z, to complete the number of parts or factors of the required co-efficient.

By this rule may be constructed the following table, in which, neglecting the derivative functions of ϕz and their divisors, which are well known, are exhibited the remaining parts or factors of the several terms in each polynomial derivative, from the first to the ninth. The parts of one derivative are deduced, according to the rule, from the corresponding parts of the preceding derivative ; and one part besides is added to each, which additional parts constitute a series of the powers of ζ.

64. TABLE II.

$\left|\begin{matrix}\varphi\alpha \\ \cdot\end{matrix}\right|^{\prime}$ ...ξ.

$\left|\begin{matrix}\varphi\alpha \\ \cdot\end{matrix}\right|^{\prime\prime}$ ...γ, ξ^2.

$\left|\begin{matrix}\varphi\alpha \\ \cdot\end{matrix}\right|^{\prime\prime\prime}$ ...δ, $2\xi\gamma$, ξ^3.

$\left|\begin{matrix}\varphi\alpha \\ \cdot\end{matrix}\right|^{\text{IV}}$ ...ε, $(2\xi\delta+\gamma^2)$, $3\xi^2\gamma$, ξ^4.

$\left|\begin{matrix}\varphi\alpha \\ \cdot\end{matrix}\right|^{\text{V}}$ ...ζ, $(2\xi\varepsilon+2\gamma\delta)$, $(3\xi^2\delta+3\xi\gamma^2)$, $4\xi^3\gamma$, ξ^5.

$\left|\begin{matrix}\varphi\alpha \\ \cdot\end{matrix}\right|^{\text{VI}}$ ...η, $(2\xi\zeta+2\gamma\varepsilon+\delta^2)$, $(3\xi^2\varepsilon+6\xi\gamma\delta+\gamma^3)$, $(4\xi^3\delta+6\xi^2\gamma^2)$, $5\xi^4\gamma$, ξ^6.

$\left|\begin{matrix}\varphi\alpha \\ \cdot\end{matrix}\right|^{\text{VII}}$ ...θ, $(2\xi\eta+2\gamma\zeta+2\delta\varepsilon)$, $(3\xi^2\zeta+6\xi\gamma\varepsilon+3\xi\delta^2+3\gamma^2\delta)$, $(4\xi^3\varepsilon+12\xi^2\gamma\delta+4\xi\gamma^3)$, $(5\xi^4\delta+10\xi^3\gamma^2)$, $6\xi^5\gamma$, ξ^7.

$\left|\begin{matrix}\varphi\alpha \\ \cdot\end{matrix}\right|^{\text{VIII}}$ ...ι, $(2\xi\theta+2\gamma\eta+2\delta\zeta+\varepsilon^2)$, $(3\xi^2\eta+6\xi\gamma\zeta+6\xi\delta\varepsilon+3\gamma^2\varepsilon+3\gamma\delta^2)$, $(4\xi^3\zeta+12\xi^2\gamma\varepsilon+6\xi^2\delta^2+12\xi\gamma^2\delta+\gamma^4)$, $(5\xi^4\varepsilon+20\xi^3\gamma\delta+10\xi^2\gamma^3)$, $(6\xi^2\delta+15\xi^4\gamma^2)$, $7\xi^6\gamma$, ξ^8.

$\left|\begin{matrix}\varphi\alpha \\ \cdot\end{matrix}\right|^{\text{IX}}$ ...$\varkappa$, $(2\xi\iota+2\gamma\theta+2\delta\eta+2\varepsilon\zeta)$, $(3\xi^2\theta+6\xi\gamma\eta+6\xi\delta\zeta+3\xi\varepsilon^2+3\gamma^2\zeta+6\gamma\delta\varepsilon+\delta^3)$, $(4\xi^3\eta+12\xi^2\gamma\zeta+12\xi^2\delta\varepsilon+12\xi\gamma^2\varepsilon+12\xi\gamma\delta^2+4\gamma^3\delta)$, $(5\xi^4\zeta+20\xi^3\gamma\varepsilon+$

$10\zeta^3\delta^2 + 30\zeta^2\gamma\delta^2 + 5\zeta\gamma^4$), ($6\zeta^5\delta + 30\zeta^4\gamma\delta + 20\zeta^3\gamma^3$), ($7\zeta^6\delta + 21\zeta^5\gamma^2$), $8\zeta^7\gamma$, ζ^8.

&c. &c.

By this table, which may be extended at pleasure, it is easy to state the value of any polynomial derivative; for we have only to multiply the several terms opposite to it, by the corresponding derivative functions of $\varphi\alpha$ with their proper divisors, and join them together by the sign of addition. Thus,

$$\left|\frac{\varphi\alpha}{\cdot}\right|^{\nu} = \varphi'\alpha.\zeta + \tfrac{1}{2}\varphi''\alpha\,(2\zeta^3 + 2\gamma\delta) + \frac{1}{2.3}\varphi'''\alpha\,(3\zeta^2\delta + 3\zeta\gamma^2) + \frac{1}{2.3.4}\varphi^{\text{IV}}\alpha.4\zeta^2\gamma + \frac{1}{2.3.4.5}\varphi^{\nu}\alpha.\zeta^5.$$

The general polynomial derivative, $\left|\frac{\varphi\alpha}{\cdot}\right|^{N}$, developed as in Article 59, may be thus expanded, putting $\frac{\varphi^N\alpha}{\cdot}$ for $\frac{1}{1.2.3\ldots n}\varphi^N\alpha$.

$$65.\ \left|\frac{\varphi\alpha}{\cdot}\right|^{N} = \frac{\varphi^N\alpha}{\cdot}\zeta^n + \frac{\varphi^{N-1}\alpha}{\cdot}\overline{n-1}.\zeta^{n-2}\gamma + \frac{\varphi^{N-2}\alpha}{\cdot}\left(\overline{n-2}\zeta^{n-3}\delta + \frac{n-2.n-3}{2}\zeta^{n-4}\gamma^2\right) + \frac{\varphi^{N-3}\alpha}{\cdot}\left(\overline{n-3}\zeta^{n-4}\zeta + \frac{n-3.n-4}{2}.\zeta^{n-5}2\gamma\delta + \frac{n-3.n-4.n-5}{2.3}\zeta^{n-6}\gamma^3\right) +$$

$$+\frac{\varphi^{N-4}\alpha}{.}\left(\overline{n-4}\varepsilon^{n-5}\zeta+\frac{n-4.n-5}{2}\varepsilon^{n-6}(2\gamma\varepsilon+\delta^2)\right.$$

$$\left.+\frac{n-4.n-5.n-6}{2.3}\varepsilon^{n-7}3\gamma^2\delta+\frac{n-4..n-7}{2.3.4}\varepsilon^{n-8}\gamma^4\right)$$

$$+\frac{\varphi^{N-5}\alpha}{.}\left(\overline{n-5}\varepsilon^{n-6}\eta+\frac{n-5.n-6}{2}\varepsilon^{n-7}(2\gamma\zeta+2\delta\varepsilon)\right.$$

$$+\frac{n-5\,.\,n-6\,.\,n-7}{2.3}\varepsilon^{n-8}(3\gamma\varepsilon^2+3\gamma\delta^2)$$

$$\left.+\frac{n-5\ldots n-8}{2.3.4}\varepsilon^{n-9}4\gamma^3\delta+\frac{n-5\ldots n-9}{2\ldots5}\varepsilon^{n-10}\gamma^5\right)$$

$$+\frac{\varphi^{N-6}\alpha}{.}\left(\overline{n-6}\varepsilon^{n-7}\theta+\frac{n-6.n-7}{2}\varepsilon^{n-8}(2\gamma\eta+2\delta\zeta+\varepsilon^2)\right.$$

$$+\frac{n-6\,..\,n-8}{2.3}\varepsilon^{n-9}(3\gamma^2\zeta+6\gamma\delta\varepsilon+\delta^3)$$

$$\left.+\frac{n-6\,..\,n-9}{2.3.4}\varepsilon^{n-10}(4\gamma^3\varepsilon+6\gamma^2\delta^2)\right.$$

$$\left.+\frac{n-6..n-10}{2..5}\varepsilon^{n-11}5\gamma^4\delta+\frac{n-6..n-11}{2\ldots6}\varepsilon^{n-12}\gamma^6\right)$$

$$+\frac{\varphi^{N-7}\alpha}{.}\left(\overline{n-7}\varepsilon^{n-8}\iota+\frac{n-7.n-8}{2}\varepsilon^{n-9}(2\gamma\theta+2\delta\eta+2\varepsilon\zeta)\right.$$

$$+\frac{n-7..n-9}{2.3}\varepsilon^{n-10}(3\gamma^2\eta+6\gamma\delta\zeta+3\gamma\varepsilon^2+3\delta^2\varepsilon)$$

$$+\frac{n-7..n-10}{2.3.4}\varepsilon^{n-11}(4\gamma^3\zeta+12\gamma^2\delta\varepsilon+4\gamma\delta^3)$$

$$+\frac{n-7\,..\,n-11}{2\,..\,5}\,\varepsilon^{n-12}\,(5\gamma^{5}\varepsilon + 10\gamma^{3}\delta^{2})$$
$$+\frac{n-7\,..\,n-12}{2\,..\,6}\varepsilon^{n-13}6\gamma^{3}\delta+\frac{n-7\,..\,n-13}{2\,..\,7}\varepsilon^{n-14}\gamma^{7}\Big)$$
$$+ \&c.$$

The law of continuation may be easily discovered, by attending to the rule for constructing Table II.

This method of developement is particularly useful, when the polynome is definite, and consists of only a small number of terms, as many of the leading terms in the derivatives of Table II. will then vanish.

To illustrate the theory in this Section, we shall now apply it to the solution of the following

PROBLEM.

66. Given the equation $x^{m} - \varepsilon x^{m-1} - \gamma x^{m-2} - \delta x^{m-3} \ldots\ldots - \lambda = o$; required the sum of the n^{th} powers of its roots, a, b, c, &c. in terms of the given coefficients, ε, γ, δ, &c.

Putting α for 1, and $x = \frac{1}{z}$, we have,

$\alpha x^m - \mathfrak{C}x^{m-1} - \gamma x^{m-2} \ldots - \lambda = \alpha.\overline{x-a}.\overline{x-b}.\overline{x-c}\ldots)$ (to m factors,)
and $\alpha - \mathfrak{C}z - \gamma z^2 \ldots - \lambda z^m = \alpha.\overline{1-az}.\overline{1-bz}.\overline{1-cz}$, &c. Hence,

$$\text{log. } (\alpha - \mathfrak{C}z - \gamma z^2 \ldots - \lambda z^m) = l.\alpha - az - \tfrac{1}{2}a^2z^2 - \tfrac{1}{3}a^3z^3 - \tfrac{1}{4}a^4z^4 - \&c.$$
$$- bz - \tfrac{1}{2}b^2z^2 - \tfrac{1}{3}b^3z^3 - \tfrac{1}{4}b^4z^4 - \&c.$$
$$- cz - \tfrac{1}{2}c^2z^2 - \tfrac{1}{3}c^3z^3 - \tfrac{1}{4}c^4z^4 - \&c.$$
$$\&c. \qquad \&c.$$

that is, putting S_1, S_2, S_3, &c. for the sums of the successive powers of the roots, a, b, c, &c. (57.)

$$l.\alpha - \left|\frac{l.\alpha}{\cdot}\right|' z - \left|\frac{l.\alpha}{\cdot}\right|'' z^2 - \left|\frac{l.\alpha}{\cdot}\right|''' z^3 - \&c.$$
$$= l.\alpha - S_1 z - \tfrac{1}{2}S_2 z^2 - \tfrac{1}{3}S_3 z^3 - \&c.$$

Hence, (58.)

$$\frac{1}{n}S_n = \left|\frac{l.\alpha}{\cdot}\right|^N = \alpha^{-1}\left|\frac{\mathfrak{C}}{\cdot}\right|^{N-1} + \tfrac{1}{2}\alpha^{-2}\left|\frac{\mathfrak{C}^2}{\cdot}\right|^{N-2} + \tfrac{1}{3}\alpha^{-3}\left|\frac{\mathfrak{C}^3}{\cdot}\right|^{N-3}$$
$$\ldots + \frac{1}{n}\alpha^{-n}\mathfrak{C}^n;$$

or, by reversing it,

$$= \frac{1}{n}\alpha^{-n}\mathfrak{C}^n + \frac{1}{n-1}\alpha^{-n+1}\left|\frac{\mathfrak{C}^{n-1}}{\cdot}\right|' + \frac{1}{n-2}\alpha^{-n+2}\left|\frac{\mathfrak{C}^{n-2}}{\cdot}\right|'' \ldots + \alpha^{-1}\left|\frac{\mathfrak{C}}{\cdot}\right|^{N-1}.$$

And, putting $\alpha = 1$, we have this general expression for S_n $(= a^n + b^n + c^n + \&c.)$ as required, viz.

$$67.\ S_n = \frac{n}{1}\left|\frac{\mathfrak{C}}{\cdot}\right|^{N-1} + \frac{n}{2}\left|\frac{\mathfrak{C}^2}{\cdot}\right|^{N-2} + \frac{n}{3}\left|\frac{\mathfrak{C}^3}{\cdot}\right|^{N-3} \ldots\ldots + \frac{n}{n}\mathfrak{C}^n.$$

$$\text{or} \quad = \mathfrak{C}^n + \frac{n}{n-1}\left|\frac{\mathfrak{C}^{n-1}}{\cdot}\right|' + \frac{n}{n-2}\left|\frac{\mathfrak{C}^{n-2}}{\cdot}\right|'' \ldots\ldots + n\left|\frac{\mathfrak{C}}{\cdot}\right|^{N-1}.$$

Thus, assuming $n = 1, 2, 3,$ &c. successively, we have the sums of the first, second, third, &c. powers of the roots as follows:

$$S_1 = \mathfrak{C}.$$
$$S_2 = \tfrac{2}{1}\gamma + \tfrac{2}{2}\mathfrak{C}^2.$$
$$S_3 = \tfrac{3}{1}\delta + \tfrac{3}{2}\cdot 2\mathfrak{C}\gamma + \tfrac{3}{3}\mathfrak{C}^3.$$
$$S_4 = \tfrac{4}{1}\varepsilon + \tfrac{4}{2}(2\mathfrak{C}\delta + \gamma^2) + \tfrac{4}{3}\cdot 3\mathfrak{C}^2\gamma + \tfrac{4}{4}\mathfrak{C}^4.$$
$$S_5 = \tfrac{5}{1}\zeta + \tfrac{5}{2}(2\mathfrak{C}\varepsilon + 2\gamma\delta) + \tfrac{5}{3}(3\mathfrak{C}^2\delta + 3\mathfrak{C}\gamma^2) + \tfrac{5}{4}\cdot 4\mathfrak{C}^3\gamma + \tfrac{5}{5}\mathfrak{C}^5.$$
&c. &c.

Which may be continued at pleasure by means of Table II.

Again, from the general equation, $l \cdot (\alpha - \mathfrak{C}z - \gamma z^2 - \text{\&c.}) = l \cdot \alpha - S_1 z - \frac{1}{2}S_2 z^2 - \frac{1}{3}S_3 z^3 - \text{\&c.}$ we have, (as in Example 1. p. 99,) $\alpha S_1 = \mathfrak{C}$; from which the succeeding terms may be obtained by derivation: then, putting $\alpha = 1$, we have,

$$S_1 = 1\mathfrak{C}.$$
$$S_2 = 2\gamma + S_1\mathfrak{C}.$$
$$S_3 = 3\delta + S_1\gamma + S_2\mathfrak{C}.$$
$$S_4 = 4\varepsilon + S_1\delta + S_2\gamma + S_3\mathfrak{C}.$$
$$S_5 = 5\zeta + S_1\varepsilon + S_2\delta + S_3\gamma + S_4\mathfrak{C}.$$
&c. &c.

Thus, for example, in the equation $x^4 - x^3 - 19x^2 + 49x - 30 = o$, of which the four roots are 1, 2, 3, and -5; we have $\epsilon = 1$, $\gamma = 19$, $\delta = -49$, and $\varepsilon = 30$. Whence,

$S_1 = 1.$	$S_4 = 723.$
$S_2 = 39.$	$S_5 = -2849.$
$S_3 = -89.$	&c. &c.

By substituting $\frac{1}{z}$ for x, we get this equation, $z^4 - \frac{49}{30}z^3 + \frac{19}{30}z^2 + \frac{1}{30}z - \frac{1}{30} = o$, whose roots are the reciprocals of those of the former; therefore, putting $\epsilon = \frac{49}{30}$, $\gamma = -\frac{19}{30}$, $\delta = -\frac{1}{30}$, and $\varepsilon = \frac{1}{30}$, we may get the sum of the $-n$ powers by the same rules.

SECTION XIV.

PRODUCT OF THE FUNCTIONS OF TWO OR MORE POLYNOMES DEVELOPED.

IT is proposed, in this Section, to develope the product of any functions whatever of two or more independent polynomes, ordered according to the powers of the same quantity z, into a series of a similar form with either, viz.

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$$

68. Taking, first, the two simple polynomes,

1. $a + bz + cz^2 + dz^3 + ez^4 + \&c.$
2. $\alpha + \zeta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \&c.$

and multiplying them together, their product is,

$a\alpha + a\zeta$	$z + a\gamma$	$z^2 + a\delta$	$z^3 + a\varepsilon$	$z^4 +$
$+ b\alpha$	$+ b\zeta$	$+ b\gamma$	$+ b\delta$	
	$+ c\alpha$	$+ c\zeta$	$+ c\gamma$	&c.
		$+ d\alpha$	$+ d\zeta$	
			$+ e\alpha$	

In this case, we have directly the required series, the co-efficients being as follows:

$$A = a\alpha.$$
$$B = a\mathfrak{C} + b\alpha.$$
$$C = a\gamma + b\mathfrak{C} + c\alpha.$$
$$D = a\delta + b\gamma + c\mathfrak{C} + d\alpha.$$
$$\&c. \qquad\qquad \&c.$$

In what manner any one of the co-efficients, B, C, D, &c. may be found is obvious: To find the fifth co-efficient E, for example, place five co-efficients of the first polynome in order, and five of the second in an inverted order; then multiply each of the former into the corresponding one of the latter, and the sum of the products is the value of E.

$a,\ b,\ c,\ d,\ e$	
$\varepsilon,\ \delta,\ \gamma,\ \mathfrak{C},\ \alpha$	
$a\varepsilon,\ b\delta,\ c\gamma,\ d\mathfrak{C},\ e\alpha$	

But it is not so plain, that the method of derivation, explained in the two preceding Sections, takes place here, and that the product of the given polynomes, (according to the notation which we have adopted,) may be thus expressed:

69. $(a+bz+cz^2+dz^3+\&c.)\times(\alpha+\mathfrak{C}z+\gamma z^2+\delta z^3+\&c.)$
$$= a\alpha + \left|\frac{a\alpha}{\cdot}\right|'.z + \left|\frac{a\alpha}{\cdot}\right|''.z^2 + \left|\frac{a\alpha}{\cdot}\right|'''.z^3 + \left|\frac{a\alpha}{\cdot}\right|^{IV}.z^4 + \&c.$$

To evince this, let it be remembered, that,

for A′, we must take B; for B′, 2C; for C′, 3D; &c.

for a', ——— b; for b', $2c$; for c', $3d$; &c.

and for α', ——— $\mathfrak{C}$; for $\mathfrak{C}'$, 2γ; for γ', 3δ; &c.

Then, since $A = a\alpha$,

$$B = (a\alpha' + \alpha a' =)\ a\mathfrak{C} + b\alpha.$$

$$2C = (a\mathfrak{C} + a'\mathfrak{C} + b'\alpha + b\alpha' =)$$

$$= 2a\gamma + b\mathfrak{C} + 2c\alpha + b\mathfrak{C};$$

therefore $C = a\gamma + b\mathfrak{C} + c\alpha$, &c.

70. In general, $\left|\frac{a\alpha}{\cdot}\right|^{N} = a\left|\frac{\alpha}{\cdot}\right|^{N} + \left|\frac{a}{\cdot}\right|'.\left|\frac{\alpha}{\cdot}\right|^{N-1} + \left|\frac{a}{\cdot}\right|''.\left|\frac{\alpha}{\cdot\cdot}\right|^{N-2}$ +&c.

Having premised this simple case of the general problem, let us now take any functions whatever of the two polynomes, as,

$F(a+bz+cz^2+dz^3+\&c.)$ and $\varphi(\alpha+\mathfrak{C}z+\gamma z^2+\delta z^3+\&c.)$

Then, if the developements of these functions be multiplied together, the leading term of the product is $Fa.\varphi\alpha$. Hence the whole product, from what has been shewn, may be expressed as follows:

71. $F(a+bz+cz^2+dz^3+\&c.)\times\varphi(\alpha+\mathfrak{C}z+\gamma z^2+\delta z^3+\&c.)$

$$= Fa.\varphi\alpha + \left|\frac{Fa.\varphi\alpha}{\cdot}\right|'.z + \left|\frac{Fa.\varphi\alpha}{\cdot}\right|''.z^2 + \left|\frac{Fa.\varphi\alpha}{\cdot}\right|'''.z^3 + \&c.$$

And for the developement of any one of these polynomial derivatives, we have the following theorem :

$$72.\ \left|\frac{Fa.\varphi\alpha}{.}\right|^{N} = Fa.\left|\frac{\varphi\alpha}{.}\right|^{N} + \left|\frac{Fa}{.}\right|'.\left|\frac{\varphi\alpha}{.}\right|^{N-1} + \left|\frac{Fa}{.}\right|''.\left|\frac{\varphi\alpha}{.}\right|^{N-2} + \ldots \left|\frac{Fa}{.}\right|^{N}.\varphi\alpha.$$

For, in multiplying the developements of the two functions together, it is evident, that the co-efficient of z^n in the product will consist of the products of every two co-efficients, one in each series, whereof the sum of the exponents of z is n; and these are the very same as those expressed in the theorem.

The propriety of expressing the product of the given functions, by the polynomial functions of its first term, will further appear from the following process of derivation :

$$A = Fa\,.\,\varphi\alpha.$$

$$B = Fa\,.\left|\frac{\varphi\alpha}{1}\right|' + \left|\frac{Fa}{1}\right|'.\,\varphi\alpha.$$

$$2\,C = Fa\,.\left|\frac{\varphi\alpha}{1}\right|'' + 2\left|\frac{Fa}{1}\right|'.\left|\frac{\varphi\alpha}{1}\right|' + \left|\frac{Fa}{1}\right|''.\varphi\alpha;\ \text{therefore,}$$

$$C = Fa\,.\left|\frac{\varphi\alpha}{1.2}\right|'' + \left|\frac{Fa}{1}\right|'.\left|\frac{\varphi\alpha}{1}\right|' + \left|\frac{Fa}{1.2}\right|.\,\varphi\alpha.$$

$$3\,D = Fa.\left|\frac{\varphi\alpha}{1.2}\right|''' + 3\left|\frac{Fa}{1}\right|'.\left|\frac{\varphi\alpha}{1.2}\right|'' + 3\left|\frac{Fa}{1.2}\right|''.\left|\frac{\varphi\alpha}{1}\right|' + \left|\frac{Fa}{1.2}\right|'''.\varphi\alpha.$$

$$D = Fa.\left|\frac{\varphi\alpha}{.}\right|''' + \left|\frac{Fa}{.}\right|'.\left|\frac{\varphi\alpha}{.}\right|'' + \left|\frac{Fa}{.}\right|''.\left|\frac{\varphi\alpha}{.}\right|' + \left|\frac{Fa}{.}\right|'''.\,\varphi\alpha.$$

&c. &c.

So that the general co-efficient A_n is rightly expressed by the series 72.

Two other expressions for the general co-efficient A_n, or $\left|\frac{Fa \,.\, \varphi\alpha}{.}\right|^N$, are given in the following theorems:

$$73.\ \left|\frac{Fa.\varphi\alpha}{.}\right|^n = Fa\left|\frac{\varphi\alpha}{.}\right|^n + F'a\left|\frac{b.\varphi\alpha}{.}\right|^{n-1} + \frac{F''a}{2}\cdot\left|\frac{b^2\,.\,\varphi\alpha}{.}\right|^{n-2} + \frac{F'''a}{2.3}\left|\frac{b^3\,.\,\varphi\alpha}{.}\right|^{n-3} \ldots\ldots\ldots\ldots + \frac{F^n a}{1\ldots n}\, b^n\,.\,\varphi\alpha.$$

$$74.\ \left|\frac{Fa.\varphi\alpha}{.}\right|^n = Fa.\varphi'\alpha\left|\frac{c}{.}\right|^{n-1} + Fa.\frac{\varphi''\alpha}{2}\left|\frac{c^2}{.}\right|^{n-2} + Fa.\frac{\varphi'''\alpha}{2.3}\left|\frac{c^3}{.}\right|^{n-3}$$
$$+ F'a.\varphi\alpha\left|\frac{b}{.}\right|^{n-1} + F'a.\varphi'\alpha\left|\frac{bc}{.}\right|^{n-2} + F'a.\frac{\varphi''\alpha}{2}\left|\frac{bc^2}{.}\right|^{n-3}$$
$$+ \frac{F''a}{2}.\varphi\alpha\left|\frac{b^2}{.}\right|^{n-2} + \frac{F''a}{2}.\varphi'\alpha\left|\frac{b^2c}{.}\right|^{n-3}$$
$$+ \frac{F'''a}{2.3}.\varphi\alpha\left|\frac{b^3}{.}\right|^{n-3} +$$
$$\&c.$$

To demonstrate these theorems, let the polynomial functions of Fa in 72. be expanded, in terms of the polynomial functions of the powers of b, and we shall have,

$$\left|\frac{Fa.\varphi\alpha}{.}\right|^n = Fa\left|\frac{\varphi\alpha}{.}\right|^n + F'a.b\left|\frac{\varphi\alpha}{.}\right|^{n-1} + \left(F'a\left|\frac{b}{.}\right|' + \frac{F''a}{2}.b^2\right).\left|\frac{\varphi\alpha}{.}\right|^{n-2}$$
$$+ \left(F'a\,.\left|\frac{b}{.}\right|'' + \frac{F''a}{2}\left|\frac{b^2}{.}\right|' + \frac{F'''a}{2.3}\,.\,b^3\right).\left|\frac{\varphi\alpha}{.}\right|^{n-3}$$
$$+ \left(F'a\left|\frac{b}{.}\right|''' + \frac{F''a}{2}\left|\frac{b^2}{.}\right|'' + \frac{F'''a}{2.3}\left|\frac{b^3}{.}\right|' + \frac{F^{IV}a}{2.3.4}.b^4\right).\left|\frac{\varphi\alpha}{.}\right|^{n-4}$$
$$+ \&c.$$

But, by 72., we have also,

$$\left|\frac{b.\varphi\alpha}{.}\right|^{n-1} = b\left|\frac{\varphi\alpha}{.}\right|^{n-1} + \left|\frac{b}{.}\right|'.\left|\frac{\varphi\alpha}{.}\right|^{n-2} + \left|\frac{b}{.}\right|''.\left|\frac{\varphi\alpha}{.}\right|^{n-3} + \left|\frac{b}{.}\right|'''.\left|\frac{\varphi\alpha}{.}\right|^{n-4} + \&c.$$

And

$$\left|\frac{b^2.\varphi\alpha}{.}\right|^{n-2} = b^2\left|\frac{\varphi\alpha}{.}\right|^{n-2} + \left|\frac{b^2}{.}\right|'.\left|\frac{\varphi\alpha}{.}\right|^{n-3} + \left|\frac{b^2}{.}\right|''.\left|\frac{\varphi\alpha}{.}\right|^{n-4} + \&c.$$

Hence the theorem in 73. is manifest.

And, by a further developement of the polynomial functions of $\varphi\alpha$, in terms of the polynomial functions of the powers of $\mathfrak{C}$, the theorem in 74. will also be evident.

75. From the last theorem, too, it is easy to see, how the general term, in the product of three or more functions of polynomes, may be developed; for, since in the first part of the developement, we have the terms of the binome $b + \mathfrak{C}$; in the second part, the terms of its second power, without numeral co-efficients, $(b^2, b\mathfrak{C}, \mathfrak{C}^2)$; in the third part, the terms of its third power, $(b^3, b^2\mathfrak{C}, b\mathfrak{C}^2, \mathfrak{C}^3,)$ &c. we have only to apply the terms (without numeral co-efficients) of the several powers of a trinome, quadrinome, &c. in the same way.

EXAMPLE 1.

$$(a + bz + cz^2 + \&c.)^r \times (\alpha + \beta z + \gamma z^2 + \&c.)^s = A + Bz + Cz^2 + Dz^3 + \&c.$$

In applying theorems 71. 72. to this example, we have,

$Fa = a^r$, $F'a = ra^{r-1}$, $F''a = r.r-1.a^{r-2}$, &c. $\varphi\alpha = \alpha^s$, &c. And the polynomial derivatives of Fa or $\varphi\alpha$ are to be found in Table II. (64.) Thus,

$A = a^r\alpha^s$.

$B = a^r . s\alpha^{s-1}\beta + \alpha^s . ra^{r-1}b$.

$$C = a^r(s\alpha^{s-1}\gamma + \frac{s.s-1}{2}\alpha^{s-2}\beta^2) + ra^{r-1}b . s\alpha^{s-1}\beta + \alpha^s(ra^{r-1}c + \frac{r.r-1}{2}a^{r-2}b^2).$$

$$D = a^r(s\alpha^{s-1}\delta + \frac{s.s-1}{2}\alpha^{s-2}.2\beta\gamma + \frac{s.s-1.s-2}{2.3}\alpha^{s-3}\beta^3) + ra^{r-1}b(s\alpha^{s-1}\gamma + \frac{s.s-1}{2}\alpha^{s-2}\beta^2) + s\alpha^{s-1}\beta(ra^{r-1}c + \frac{r.r-1}{2}a^{r-2}b^2) + \alpha^s(ra^{r-1}d + \frac{r.r-1}{2}a^{r-2}2bc + \frac{r.r-1.r-2}{2.3}a^{r-3}b^3).$$

$$E = a^r(s\alpha^{s-1}\varepsilon + \frac{s.s-1}{2}\alpha^{s-2}(2\beta\delta + \gamma^2) + \frac{s.s-1.s-2}{2.3}\alpha^{s-3}.3\beta^2\gamma$$

$$+\frac{s\ldots s-3}{2\ldots4}\alpha^{s-4}.\zeta^4)+ra^{r-1}b(s\alpha^{s-1}\delta+\frac{s.s-1}{2}\alpha^{s-2}.2\zeta\gamma$$

$$+\frac{s.s-1.s-2}{2.3}\alpha^{s-3}.\zeta^3)+(ra^{r-1}c+\frac{r.r-1}{2}a^{r-2}.b^2)$$

$$.(s\alpha^{s-1}\gamma+\frac{s.s-1}{2}\alpha^{s-2}.\zeta^2)+(ra^{r-1}d+\frac{r.r-1}{2}a^{r-2}.2bc$$

$$+\frac{r.r-1.r-2}{2.3}a^{r-3}.b^3).s\alpha^{s-1}.\zeta+(ra^{r-1}e$$

$$+\frac{r.r-1}{2}a^{r-2}(2bd+c^2)+\frac{r.r-1.r-2}{2.3}a^{r-3}.3b^2c$$

$$+\frac{r\ldots r-3}{2.3.4}a^{r-4}b^4)\alpha^s+\&c.$$

See pp. 54, 55, 56.

EXAMPLE 2.

$$\frac{(\alpha+\zeta z+\gamma z^2+\delta z^3+\&c.)^s}{(a+bz+cz^2+dz^3+\&c.)^r}=A+Bz+Cz^2+Dz^3+\&c.$$

Here we have only to change $+r$, in the preceding example, into $-r$, and we have,

$$A=a^{-r}\alpha^s.$$

$$B=a^{-r}.s\alpha^{s-1}\zeta-\alpha^s.ra^{-r-1}b.$$

&c. &c.

EXAMPLE 3.

To convert the fraction, $\frac{\alpha x + \epsilon}{ax^2 + bx + c}$, into a series of this form, $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \&c.$

By putting $z = \frac{1}{x}$, we shall have,

$$\frac{\alpha + \epsilon z}{a + bz + cz^2} = A + Bz + Cz^2 + \&c.$$

a case of the preceding example, in which $s = 1$, $r = -1$; γ, δ, &c. $= o$; d, e, &c. $= o$.

Hence $A = a^{-1}\alpha$.

$B = a^{-1}\epsilon - a^{-2}.ab.$

$C = -a^{-2}b\epsilon + a^{-3}ab^2 - a^{-2}\alpha c.$

$D = -a^{-2}c\epsilon + a^{-3}(b^2\epsilon + 2abc) - a^{-4}.\alpha b^3.$

&c. &c.

SECTION XV.

OF RECURRING SERIES.

LET $A, A_1, A_2, A_3, A_4 \ldots\ldots A_m, A_{m+1}, A_{m+2} \ldots\ldots$ denote any recurring series of the n^{th} order, that is, a series of which, taking any term A_m, and n terms immediately following, and multiplying each in order by some given quantity, (affirmative or negative,) the sum of the several products shall vanish. Thus,

$$\alpha A_m + \beta A_{m+1} + \gamma A_{m+2} + \delta A_{m+3} \ldots\ldots + \alpha_n . A_{m+n} = o.$$

It is proposed to find an expression for the general term of such a series; also, to find the sum of the series continued indefinitely, and the sum of a given number of terms.

To simplify the investigation, we shall confine our attention to a series of the third order, of which the equation is,

$$\alpha A_m + \beta A_{m+1} + \gamma A_{m+2} + \delta A_{m+3} = o.$$

Now, considering A_1, A_2, A_3, &c. as polynomial derivatives of A, and $\mathfrak{C}$, γ, δ as polynomial derivatives of α, this equation, according to the notation we have adopted, becomes,

$$\alpha \left|\frac{A}{\cdot}\right|^m + \left|\frac{\alpha}{\cdot}\right|' \cdot \left|\frac{A}{\cdot}\right|^{m-1} + \left|\frac{\alpha}{\cdot}\right|'' \cdot \left|\frac{A}{\cdot}\right|^{m-2} + \left|\frac{\alpha}{\cdot}\right|''' \cdot \left|\frac{A}{\cdot}\right|^{m-3} = o.$$

one side of which is the developement of $\left|\frac{\alpha A}{\cdot}\right|^m$, when $\left|\frac{\alpha}{\cdot}\right|^{IV}$, $\left|\frac{\alpha}{\cdot}\right|^V$, &c. vanish, and is therefore equivalent to the co-efficient of x^m, in the developement of the product,

$(\alpha + \mathfrak{C}x + \gamma x^2 + \delta x^3) \times (A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.)$

where the first polynome is limited to four terms, and the second is indefinite.

Let that product be expressed by the series,

$$a + bx + cx^2 + dx^3 + ex^4 + \&c.$$

then we shall have,

$a = \alpha A$. And by derivation, (69.)

$b = \alpha B + \mathfrak{C}A = \alpha A^1 + \mathfrak{C}A.$

$c = \alpha C + \mathfrak{C}B + \gamma A = \alpha A_2 + \mathfrak{C}A_1 + \gamma A.$

$d = \alpha A_3 + \mathfrak{C}A_2 + \gamma A_1 + \delta A = o$, by hypothesis,

$e = \alpha A_4 + \mathfrak{C}A_3 + \gamma A_2^1 + \delta A = o.$

&c. &c.

Hence the series, expressing the said product, is limited to three terms, viz. $a+bx+cx^2$: And $\frac{a+bx+cx^2}{\alpha+\beta x+\gamma x^2+\delta x^3}$ ($= A + Bx + Cx^2 + Dx^3 +$ &c.) is termed the generating fraction of the recurring series; which fraction has given co-efficients in the numerator, (as well as in the denominator,) when the first three terms of the recurring series are given.

The first part of the problem, then, is reduced to this:—To find the co-efficient of x_m in the developement of the generating fraction; for that is equivalent to A_m, the general term of the series whose equation is given.

So that this part of the problem has been already resolved, the general term having been found in the preceding Section to be,

$$A_m = \left|\frac{a\alpha^{-1}}{\cdot}\right|^m = a\left|\frac{\alpha^{-1}}{\cdot}\right|^m + b\left|\frac{\alpha^{-1}}{\cdot}\right|^{m-1} + c\left|\frac{\alpha^{-1}}{\cdot}\right|^{m-2},$$

where $a=\alpha A$, $b=\alpha A_1+\beta A$, and $c=\alpha A_2+\beta A_1+\gamma A$. And the polynomial derivatives of α^{-1} are to be found in Table II.

With respect to the other parts of the problem, it is plain, 1. That the generating fraction, when $x = 1$, is the sum of the recurring series continued indefinitely, provided neither of its terms vanish by putting $x = 1$,

and the series be such as can be summed. Or, if both terms of the fraction vanish, by making $x = 1$, then its value, (that is, the sum of the series,) may be found, by taking the derivative functions of the numerator and denominator, as we have before shewn. 2. That to find the sum of a definite number of terms, we have only to add to the numerator of the generating fraction, those concluding terms of the series a, bx, cx^2, &c. which, in consequence of the termination of the given series A, Bx, Cx^2, &c. do not vanish.

If, for example, the sum of $\overline{r+1}$ terms of a recurring series of the third order be required, we have,

$$\begin{aligned}
a &= \alpha A.\\
b &= \alpha A_1 + \mathfrak{C} A.\\
c &= \alpha A_2 + \mathfrak{C} A_1 + \gamma A.\\
d &= \alpha A_3 + \mathfrak{C} A_2 + \gamma A_1 + \delta A = o. \quad \text{Hence,}\\
e &= o,\ f = o, \text{ \&c. \&c. And}\\
a_r &= \alpha A_r + \mathfrak{C} A_{r-1} + \gamma A_{r-2} + \delta A_{r-3} = o. \quad \text{But}\\
a_{r+1} &= * \quad \mathfrak{C} A_r + \gamma A_{r-1} + \delta A_{r-2}.\\
a_{r+2} &= * \quad * \quad \gamma A_r + \delta A_{r-1}.\\
a_{r+3} &= * \quad * \quad * \quad \delta A_r.
\end{aligned}$$

Whence the generating fraction, expressing the sum of $\overline{r+1}$ terms of the series, when $x = 1$, will be,

$$\frac{a + bx + cx^2 + a_{r+1}x^{r+1} + a_{r+2}x^{r+2} + a_{r+3}x^{r+3}}{\alpha + \mathfrak{C}x + \gamma x^2 + \delta x^3}$$

What has been said of a recurring series of the third order, may be easily applied to one of the n^{th} order.

Thus, the equation of a series of the n^{th} order is

$$\alpha A_m + \mathfrak{C} A_{m-1} + \gamma A_{m-2} + \ldots\ldots + \alpha_n A_{m-n} = o.$$

And the generating fraction as follows,

$$\frac{a + bx + cx^2 + dx^3 + \ldots\ldots + a_{n-1}x^{n-1}}{\alpha + \mathfrak{C}x + \gamma x^2 + \delta x^3 + \ldots\ldots + \alpha_n x^n},$$

there being n terms in the numerator, and $\overline{n+1}$ terms in the denominator; and the co-efficients in the former are given when n terms of the series are given.

Also, the general term of the series is thus expressed,

$$A_m = \left|\frac{a\alpha^{-1}}{.}\right|^m = a\left|\frac{\alpha^{-1}}{.}\right|^m + b\left|\frac{\alpha^{-1}}{.}\right|^{m-1} + c\left|\frac{\alpha^{-1}}{.}\right|^{m-2} + \ldots + a_{n-1}\left|\frac{\alpha^{-1}}{.}\right|^{m-n+1}.$$

The generating fraction expresses the sum of the series, $A + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \&c.$; therefore, by putting $x = 1$, it will then express the sum of $A + A_1 + A_2 + A_3 + \&c.$, provided this be a converging series that admits of being summed.

Lastly, to express the sum of $\overline{r+1}$ terms of the recurring series, the generating fraction will be thus changed,

$$\frac{a + bx + \ldots + a_{n-1}x^{n-1} + a_{r+1}x^{r+1} + a_{r+2}x^{r+2} + \ldots + a_{r+n}x^{r+n}}{\alpha + \mathfrak{C}x + \gamma x^2 + \delta x^3 + \ldots\ldots + \alpha_n x^n}.$$

EXAMPLE 1.

To find the sum of n terms of the series of square numbers, 1, 4, 9, 16, 25, &c.

Let the given series be thus expressed,

$$0,\ 1,\ 4,\ 9,\ 16,\ \ldots\ldots\ n^2.$$
$$A,\ A_1,\ A_2,\ A_3,\ A_4,\ \ldots\ldots\ A_n.$$

Then, because the second differences are equal, the third order of differences $= o$; hence this equation,

$$A_n - 3A_{n-1} + 3A_{n-2} - A_{n-3} = o,$$

where A_n denotes the general term of the series. We have therefore a recurring series of the third order, and the given co-efficients of its equation are,

$$\alpha = 1,\ \mathfrak{C} = -3,\ \gamma = 3,\ \text{and}\ \delta = -1.$$

Hence the first three terms in the numerator of the generating fraction are $a = \alpha A = o$.

$$b = \alpha A_1 + \mathfrak{C} A = 1.$$
$$c = \alpha A_2 + \mathfrak{C} A_1 + \gamma A = 1.$$

And the last three terms of the same are,

$$a_{n+1} = \mathfrak{C} A_n + \gamma A_{n-1} + \delta A_{n-2} = -A_{n+1}.$$
$$a_{n+2} = \gamma A_n + \delta A_{n-1} = 3A_n - A_{n-1}.$$
$$a_{n+3} = \delta A_n = -A_n.$$

Thus, the generating fraction is

$$\frac{o + x + x^2 - A_{n+1}x^{n+1} + (3A_n - A_{n-1})x_{n+2} - A_n x^{n+3}}{(1-x)^3},$$

which, when $x = 1$, gives the value of S, the required sum.

But, in that case, the numerator as well as the denominator vanishes, for $2 - A_{n+1} + 2A_n - A_{n-1} = o$, because 2 is the second difference of the given series. Taking, therefore, the third derivative of each term, and then putting $x = 1$, we have

$$S = \frac{-\overline{n+1}.n.\overline{n-1}.A_{n+1} + \overline{n+2}.\overline{n+1}.n(3A_n - A_{n-1}) - \overline{n+3}.\overline{n+2}.\overline{n+1}.A_n}{-2.3};$$

or,

$$S = \frac{\overline{n+1}.n.n-1}{2.3}A_{n+1} - \frac{\overline{n+2}.\overline{n+1}.2n-3}{2.3}A_n + \frac{\overline{n+2}.\overline{n+1}.n}{2.3}A_{n-1},$$

in which, substituting $(n+1)^2$ for A_{n+1}, n^2 for A_n, &c. we get $S = \frac{n.\overline{n+1}.2n+1}{2.3}$.

EXAMPLE 2.

Required the sums of the following series:

$1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 +$ &c.

$1 + 12x + 42x^2 + 100x^3 + 195x^4 + 336x^5 +$ &c.

$1 + 48x + 366x^2 + 1360x^3 + 3615x^4 + 7896x^5 +$ &c.

the second differences of the co-efficients in the first series, the third differences in the second, and the fourth differences in the third, being equal.

It appears from the preceding example, that when the n^{th} order of differences $= o$, the generating fraction is,

$$\frac{a + bx + cx^2 + \ldots + a_{n-1}x^{n-1}}{(1-x)}.$$

Since, therefore, $\alpha = 1$, $\mathfrak{C} = -n$, $\gamma = \frac{n \cdot n-1}{2}$, $d = -\frac{n.n-1.n-2}{2.3}$, &c. we shall have,

$$a = \alpha A = 1.$$

$$b = \alpha A_1 + \mathfrak{C} A = A_1 - n.$$

$$c = \alpha A_2 + \mathfrak{C} A_1 + \gamma A = A_2 - nA_1 + \frac{n.n-1}{2}.$$

$$d = A_3 - nA_2 + \frac{n.n-1}{2} A_1 - \frac{n.n-1.n-2}{2.3}.$$

&c. &c.

The law of continuation is manifest. Thus, for the third series, we have $S = \frac{1 + 43x + 136x^2}{(1-x)^3}$.

EXAMPLE 3.

Given this recurring series of the second order, viz.

Sin. o, sin. t, sin. $2t$, sin. $3t$ . . . . sin. nt . . . .

of which the equation is sin. $nt - 2$ cos. t. sin. $\overline{n-1}.t$ + sin. $\overline{n-2}.t = o$; it is required to find the value of (sin. nt) the general term of the series; also, to find the sum of the series continued indefinitely, and the sum of $\overline{n+1}$ terms of the series.

Here $\alpha = 1$, $\mathfrak{C} = -2\cos. t$, and $\gamma = 1$.

Also, $a = \alpha A = o$, $b = \alpha A_1 + \mathfrak{C}A = \sin. t$; therefore

$A_n = a\left|\frac{\alpha^{-1}}{\cdot}\right|^n + b\left|\frac{\alpha^{-1}}{\cdot}\right|^{n-1} = (\text{since } a = o)\, b\left|\frac{\alpha^{-1}}{\cdot}\right|^{n-1}$, which developed by 65, (observing that, since $\alpha = 1$, every odd derivative of α^{-1} with its proper divisor $= -1$, and every even derivative with its proper divisor $= +1$: also, since $\mathfrak{C}$ is negative, every odd power of $\mathfrak{C}$ is negative, and every even power affirmative,) gives

$$A_n = b\left(\mathfrak{C}^{n-1} - \overline{n-2}.\mathfrak{C}^{n-3}.\gamma + \frac{n-3.n-4}{2}\mathfrak{C}^{n-5}.\gamma^2 - \frac{n-4.n-5.n-6}{2.3}\right.$$
$$\left.\mathfrak{C}^{n-7}.\gamma^3 + \right), \text{ \&c.}$$

in which, substituting for A_n, b, $\mathfrak{C}$ and γ, their respective values, viz. sin. nt, sin. t, $2\cos. t$ $(= C)$ and 1, we have

$$\sin. nt = \sin. t\left(C^{n-1} - \overline{n-2}.C^{n-3} + \frac{n-3.n-4}{2}C^{n-5} - \right.$$
$$\left.\frac{n-4.n-5.n-6}{2.3}C^{n-7} + \text{\&c.}\right)$$

an expression for the general term of the series.

Next, the generating fraction, viz.

$$\frac{a + bx}{\alpha + \mathfrak{C}x + \gamma x^2} = \frac{\sin. t\,.\,x}{1 - 2\cos. t\,.\,x + x^2},$$

gives, when $x = 1$, $\frac{\sin. t}{2 - 2\cos. t} = \frac{1}{2}\cot. \frac{1}{2}t$ for the sum of the series continued indefinitely.

Lastly, for a part of the series, (sin. nt being the last term,) we have

$$\frac{a+bx+a_{n+1}x^{n+1}+a_{n+2}x^{n+2}}{\alpha+\beta x+\gamma x^2} = \frac{\sin. t.x - \alpha A_{n+1}x^{n+1}+\gamma A_n x^{n+2}}{1-2\cos. t.x+x^2}$$

for the generating fraction, which, when $x = 1$, gives the sum required,

$$= \frac{\sin. t - \sin.\overline{n+1}.t + \sin. nt}{2 \text{ ver. } t} = \frac{\sin.\frac{n+1}{2} . t . \sin. \frac{1}{2}nt}{\sin. \frac{1}{2}t}.$$

Cor. 1. Hence, if $nt = 180°$, then $\sin.\overline{n+1} . t = \sin.\overline{180+t} = -\sin. t$, and $\sin. nt = o$; the sum of the sines therefore to $180° = \frac{2 \sin. t}{2 \text{ ver. } t} = \cot. \frac{1}{2}t . n$ being any whole number.

Cor. 2. If $nt = 90°$, the sum of the sines of a quadrant $= \frac{1}{2} + \frac{1}{2}\cot. \frac{1}{2}t$.

EXAMPLE 4.

Required the sum of the following series,

$$x - 5x^3 + 14x^5 - 30x^7 + 55x^9 - \&c.$$

where the third differences of the co-efficients are equal.

Let the given series be expressed thus,

$$A + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \&c.;$$

then, since the fourth differences of the co-efficients vanish, by putting $x = 1$, we get a recurring series of the fourth order, of which the equation is

$$A + 4A_1 + 6A_2 + 4A_3 + A_4 = o;$$

hence $\alpha = 1$, $\beta = 4$, $\gamma = 6$, $\delta = 4$, and $\varepsilon = 1$.

Now, assuming $\alpha + \beta x^2 + \gamma x^4 + \delta x^6 + \varepsilon x^8$ for the denominator of the generating fraction, in order that its product with the given series (that is, the numerator,) may be expressed by $a + bx + cx^2 + dx^3$, a series similar to the given one; we have then

$$a = \alpha A = o.$$
$$b = \alpha A_1 + \beta A = 1.$$
$$c = \alpha A_2 + \beta A_1 + \gamma A = -1.$$
$$d = \alpha A_3 + \beta A_2 + \gamma A_1 + \delta A = o.$$

Hence, the generating fraction, or sum of the given series, is $\frac{x - x^3}{(1 + x^2)^4}$.

EXAMPLE 5.

Given the series $1 - 15x^2 + 70x^4 - 210x^6 + 495x^8 - 1001x^{10}$ + &c. to be summed, the fourth differences of the co-efficients being equal.

Given series, ----- $A + A_1x^2 + A_2x^4 + A_3x^6 + A_4x^8 +$ &c.

Denom. of gen. fr. $\alpha + \beta x^2 + \gamma x^4 + \delta x^6 + \varepsilon x^8 + \zeta x^{10}$.

And the numerator, - - - $a + bx^2 + cx^4 + dx^6 + ex^8$.
But, $\alpha = 1$, $\mathfrak{G} = 5$, $\gamma = 10$, $\delta = 10$, $\mathfrak{s} = 5$, and $\zeta = 1$.
And $A = 1$, $A_1 = -15$, $A_2 = 70$, $A_3 = -210$, &c.
Hence, $a = 1$, $b = -10$, $c = 5$, $d = o$, and $e = o$.
Therefore the generating fraction $= \frac{1 - 10x^2 + 5x^4}{(1 + x^2)^5}$.

EXAMPLE 6.

To find the sum of $1 + mx + \overline{m+1}.x^2 + \overline{m+2}.x^3 + \overline{m+3}.x^4 +$ &c.

Omitting the first term, and putting $x = 1$, we have a recurring series of the second order, whose equation is,

$$A - 2A_1 + A_2 = o.$$

Hence, $\alpha = 1$, $\mathfrak{G} = -2$, and $\gamma = 1$; also, $a = m$, and $b = -\overline{m-1}$.

And the generating fraction $= \frac{mx - \overline{m-1}.x^2}{(1-x)^2}$.

Consequently $1 + \frac{mx - \overline{m-1}.x^2}{(1-x)^2} = \frac{1 + \overline{m-2}.x - \overline{m-2}.x^2}{(1-x)^2}$ = sum required.

EXAMPLE 7.

To find the sum of $1 + (3x)^2 + (5x^2)^2 + (7x^3)^2 + (9x^4)^2 +$ &c.

The generating fraction may be found, in like manner, for this series, to be $\frac{1 + 6x^2 + x^4}{(1-x)^3}$.

EXAMPLE 8.

To sum $\frac{1^n}{x} + \frac{2^n}{x^2} + \frac{3^n}{x^3} + \frac{4^n}{x^4} + \frac{5^n}{x^5} +$ &c.

This, when $x = 1$, is a recurring series of the $\overline{n+1}$ order, whose equation is

$$.. A - \overline{n+1}.A_1 + \frac{n+1.n}{2}A_2 - \frac{n+1.n.n-1}{2.3}A_3 + \&c. = o,$$

so that $\alpha = 1$, $6 = -\overline{n+1}$, $\gamma = \frac{n+1.n}{2}$, $\delta = -\frac{n+1.n.n-1}{2.3}$, &c.

Hence,

$$a = o,\ b = 1,\ c = 2^n - \overline{n+1},\ d = 3^n - \overline{n+1}.2^n + \frac{n+1.n}{2},$$

$$e = 4^n - \overline{n+1} \cdot 3^n + \frac{n+1 \cdot n}{2}2^n - \frac{n+1.n.n-1}{2 \cdot 3},$$

&c.; therefore, putting $\frac{1}{x} = y$,

$$\frac{a+by+cy^2+dy^3+}{(1-y)^{n+1}} \&c. = y + 2^n.y^2 + 3^n.y^3 + 4^n.y^4 + 5^n.y^5 + \&c.,$$

that is,

$$\frac{y+\left(2^n-\overline{n+1}\right)y^2+\left(3^n-\overline{n+1}.2^n+\frac{n+1.n}{2}\right)y^3+\left(4^n-\overline{n+1}.3^n+\frac{n+1.n}{2}2^n-\frac{n+1.n.n-1}{2.3}\right)}{(1-y)^{n+1}}$$

or,

$$\frac{x^n+\left(2^n-\overline{n+1}\right)x^{n-1}+\left(3^n-\overline{n+1}.2^n+\frac{n+1.n}{2}\right)x^{n-2}+\text{ \&c.}}{(x-1)^{n+1}}=\text{sum}$$

required.

Cor. Assuming $n=1$, 2, 3, &c. successively, we have,

$$\frac{y}{(1-y)^2} \quad =y+2y^2+3y^3+4y^4+5y^5+\text{ \&c.}$$

$$\frac{y+y^2}{(1-y)^3} \quad =y+2^2y^2+3^2y^3+4^2y^4+5^2y^5+\text{ \&c.}$$

$$\frac{y+4y^2+y^3}{(1-y)^4} \quad =y+2^3y^2+3^3y^3+4^3y^4+5^3y^5+\text{ \&c.}$$

$$\frac{y+11y^2+11y^3+y^4}{(1-y)^5} \quad =y+2^4y^2+3^4y^3+4^4y^4+5^4y^5+\text{ \&c.}$$

$$\frac{y+26y^2+66y^3+26y^4+y^5}{(1-y)^6} \quad =y+2^5y^2+3^5y^3+4^5y^4+5^5y^5+\text{ \&c.}$$

$$\frac{y+5y^2+302y^3+302y^4+57y^5+y^6}{(1-y)^7} \quad =y+2^6y^2+3^6y^3+4^6y^4+5^6y^5+\text{ \&c.}$$

&c. &c.

The numerators of these fractions may be deduced, each from the preceding. Thus, let $y^n+ay^{n-1}+by^{n-2}+cy^{n-3}+$ &c. be a numerator, then the next following will be,

$$y^{n+1}+(n+2a)y^n+(\overline{n-1}.a+3b)y^{n-1}+(\overline{n-2}.b+4c)y^{n-2}+(\overline{n-3}.c+5d)y^{n-3}+\text{ \&c.} \qquad \text{\&c.}$$

SECTION XVI.

PROPERTIES OF POLYNOMIAL FUNCTIONS.

THE nature and properties of polynomial functions will be farther illustrated by the following additional theorems, to which we shall afterwards have occasion to refer.

76. THEOREM 1.

$\frac{r}{n+r}\left|\frac{\alpha^{n+r}}{.}\right|' = \alpha^n \left|\frac{\alpha^r}{.}\right|'$. For $r\alpha^{n+r-1}.\xi$ is the developement of each side of the equation.

77. THEOREM 2.

$\left|\frac{\alpha^n}{.}\right|' \times \left|\frac{\alpha^r}{.}\right|' = nr.\alpha^{n+r-2}.\xi^2$. For this is the product of the developements of the two functions.

78. THEOREM 3.

$$\frac{r}{n+r}\left|\frac{\alpha^{n+r}}{.}\right|'' = \alpha^n \left|\frac{\alpha^r}{.}\right|'' + \tfrac{1}{2}\left|\frac{\alpha^n}{.}\right|' . \left|\frac{\alpha^r}{.}\right|',$$

$$= -\alpha^{n+2r}\left|\frac{\alpha^{-r}}{.}\right|'' - \tfrac{1}{2}\alpha^{2n+2r}\left|\frac{\alpha^{-n-2r}}{.}\right|' . \left|\frac{\alpha^r}{.}\right|'.$$

For $\alpha^n \left|\frac{\alpha_r}{.}\right|'' = r\alpha^{n+r-1}.\gamma + \frac{r.r-1}{2}\alpha^{n+r-2}.\epsilon^2$,

and $\frac{1}{2}\left|\frac{\alpha^n}{.}\right|'.\left|\frac{\alpha^r}{.}\right|' = \frac{rn}{2}\alpha^{n+r-2}.\epsilon^2$. (77.)

the sum of which $r\alpha^{n+r-1}.\gamma + \frac{r.n+r-1}{2}\alpha^{n+r-2}\epsilon^2$ is the developement of the first side of the equation; the other part of the theorem may be shewn in the same manner.

79. THEOREM 4.

$\frac{r}{n+r}\left|\frac{\alpha^{n+r}}{.}\right|''' = \frac{3}{3}\alpha^n\left|\frac{\alpha^r}{.}\right|''' + \frac{2}{3}\left|\frac{\alpha^n}{.}\right|'.\left|\frac{\alpha^r}{.}\right|'' + \frac{1}{3}\left|\frac{\alpha^n}{.}\right|''.\left|\frac{\alpha^r}{.}\right|'$, as appears in like manner: And so on, for any higher function. In general, therefore,

80. THEOREM 5.

$$\frac{r}{n+r}\left|\frac{\alpha^{n+r}}{.}\right|^m = \frac{m}{m}\alpha^n\left|\frac{\alpha^r}{.}\right|^m + \frac{m-1}{m}\left|\frac{\alpha^n}{.}\right|'.\left|\frac{\alpha^r}{.}\right|^{m-1} + \frac{m-2}{m}\left|\frac{\alpha^n}{.}\right|''.\left|\frac{\alpha^r}{.}\right|^{m-2}$$
$$+ \frac{m-3}{m}\left|\frac{\alpha^n}{.}\right|'''.\left|\frac{\alpha^r}{.}\right|^{m-3} + \ldots\ldots\ldots + \left|\frac{\alpha^n}{.}\right|^{m-1}.\left|\frac{\alpha^r}{.}\right|'$$

81. THEOREM 6.

$$\rho = \frac{1}{m}\left|\frac{\alpha^{-m}}{.}\right|^0.\left|\frac{\alpha^m}{.}\right|^n + \frac{1}{m+\partial}\left|\frac{\alpha^{-m-\partial}}{.}\right|'.\left|\frac{\alpha^{m+\partial}}{.}\right|^{n-1}$$
$$+ \frac{1}{m+2\partial}\left|\frac{\alpha^{-m-2\partial}}{.}\right|''.\left|\frac{\alpha^{m+2\partial}}{.}\right|^{n-2} + \ldots\ldots\ldots$$
$$+ \frac{1}{m+n\partial}\left|\frac{\alpha^{-m-n\partial}}{.}\right|^n.\left|\frac{\alpha^{m+n\partial}}{.}\right|^0.$$

This theorem may be proved, like the preceding one, by induction. Thus, putting $m + \partial = r$, $m + 2\partial = t$, $m + 3\partial = v$, &c. then, first, taking $n = 1$, we have,

$$o = \frac{1}{m}\left|\frac{\alpha^{-m}}{.}\right|^{o}.\left|\frac{\alpha^{m}}{.}\right|' + \frac{1}{r}\left|\frac{\alpha^{-r}}{.}\right|'.\left|\frac{\alpha^{r}}{.}\right|^{o} = \frac{1}{m}\alpha^{-m}\left|\frac{\alpha^{m}}{.}\right|' + \frac{1}{r}\left|\frac{\alpha^{-r}}{.}\right|'.\alpha^{r}$$

$$= \alpha^{-1}\mathfrak{C} - \alpha^{-1}\mathfrak{C}.$$

Next, taking $n = 2$,

$$o = \frac{1}{m}\left|\frac{\alpha^{-m}}{.}\right|^{o}.\left|\frac{\alpha^{m}}{.}\right|'' + \frac{1}{r}\left|\frac{\alpha^{-r}}{.}\right|'.\left|\frac{\alpha^{r}}{.}\right|' + \frac{1}{t}\left|\frac{\alpha^{-t}}{.}\right|''.\left|\frac{\alpha^{t}}{.}\right|^{o} = \alpha^{-1}\gamma$$

$$+ \frac{m-1}{2}.\alpha^{-2}.\mathfrak{C}^{2} - r.\alpha^{-2}\mathfrak{C}^{2} - \alpha^{-1}\gamma + \frac{t+1}{2}.\alpha^{-2}\mathfrak{C}^{2}; \text{ be-}$$

cause $2r = \overline{m + t}$.

Again, taking $n = 3$,

$$\frac{1}{m}\left|\frac{\alpha^{-m}}{.}\right|^{o}.\left|\frac{\alpha^{m}}{.}\right|''' + \frac{1}{r}\left|\frac{\alpha^{-r}}{.}\right|'.\left|\frac{\alpha^{r}}{.}\right|'' + \frac{1}{t}\left|\frac{\alpha^{-t}}{.}\right|''.\left|\frac{\alpha^{t}}{.}\right|' + \frac{1}{v}\left|\frac{\alpha^{-v}}{.}\right|'.\left|\frac{\alpha^{v}}{.}\right|^{o}$$

$$= \begin{cases} \alpha^{-1}\delta + \dfrac{m-1}{2}\,\alpha^{-2}.2\mathfrak{C}\gamma + \dfrac{m-1.m-2}{2.3}\,\alpha^{-3}\mathfrak{C}^{3}. \\ \quad - \dfrac{r}{2}\,\alpha^{-2}.2\mathfrak{C}\gamma - \dfrac{r.r-1}{2}\,\alpha^{-3}\mathfrak{C}^{3}. \\ \quad - \dfrac{t}{2}\,\alpha^{-2}.2\mathfrak{C}\gamma + \dfrac{t.t+1}{2}\,\alpha^{-3}.\mathfrak{C}^{3}. \\ -\alpha^{-1}\delta + \dfrac{v+1}{2}\,\alpha^{-2}.2\mathfrak{C}\gamma - \dfrac{v+1.v+2}{2.3}\,\alpha^{-3}.\mathfrak{C}^{3}. \end{cases}$$

$= o$, because $\overline{m + v} = \overline{r + t}$, and $\overline{r + v} = 2t$.

&c. &c.

82. THEOREM 7.

$$\left|\frac{\varepsilon^r}{\cdot}\right|^{n-r} = \left|\frac{\alpha^r}{\cdot}\right|^n - r\alpha\left|\frac{\alpha^{r-1}}{\cdot}\right|^n + \frac{r.r-1}{2}\alpha^2\left|\frac{\alpha^{r-2}}{\cdot}\right|^n - \frac{r.r-1.r-2}{2.3}\alpha^3\left|\frac{\alpha^{r-3}}{\cdot}\right|^n$$

$$+ \ldots\ldots\ldots \pm r\alpha^{r-1}.\left|\frac{\alpha}{\cdot}\right|^n, \text{ if } n > r.$$

For, since $\left|\frac{\alpha^2}{\cdot}\right|^n = 2\alpha\left|\frac{\varepsilon}{\cdot}\right|^{n-1} + \left|\frac{\varepsilon^2}{\cdot}\right|^{n-2}$, and $\left|\frac{\varepsilon}{\cdot}\right|^{n-1} = \left|\frac{\alpha}{\cdot}\right|^n$, (60.)

therefore, $\left|\frac{\varepsilon^2}{\cdot}\right|^{n-2} = \left|\frac{\alpha^2}{\cdot}\right|^n - 2\alpha\left|\frac{\alpha}{\cdot}\right|^n$.

Again, because $\left|\frac{\alpha^2}{\cdot}\right|^n = 3\alpha^2\left|\frac{\varepsilon}{\cdot}\right|^{n-1} + 3\alpha\left|\frac{\varepsilon^2}{\cdot}\right|^{n-2} + \left|\frac{\varepsilon^3}{\cdot}\right|^{n-3}$

$$= 3\alpha^2\left|\frac{\alpha}{\cdot}\right|^n + 3\alpha\left|\frac{\alpha^2}{\cdot}\right|^n - 6\alpha^3\left|\frac{\alpha}{\cdot}\right|^n + \left|\frac{\varepsilon^3}{\cdot}\right|^{n-3};$$

therefore, $\left|\frac{\varepsilon^3}{\cdot}\right|^{n-3} = \left|\frac{\alpha^3}{\cdot}\right|^n - 3\alpha\left|\frac{\alpha^2}{\cdot}\right|^n + 3\alpha^2\left|\frac{\alpha}{\cdot}\right|^n$. In like manner, $\left|\frac{\varepsilon^4}{\cdot}\right|^{n-4} = \left|\frac{\alpha^4}{\cdot}\right|^n - 4\alpha\left|\frac{\alpha^3}{\cdot}\right|^n + 6\alpha^2\left|\frac{\alpha^2}{\cdot}\right|^n - 4\alpha^3\left|\frac{\alpha}{\cdot}\right|^n$.

&c. &c.

Hence a polynomial derivative may be developed in terms of the leading quantity of the polynome. Thus,

83. THEOREM 8.

$$\left|\frac{\varphi\alpha}{\cdot}\right|^n = \varphi'\alpha\left|\frac{\alpha}{\cdot}\right|^n + \tfrac{1}{2}\varphi''\alpha\left(\left|\frac{\alpha^2}{\cdot}\right|^n - 2\alpha\left|\frac{\alpha}{\cdot}\right|^n\right) + \frac{1}{2.3}\varphi'''\alpha.$$

$$\left(\left|\frac{\alpha^3}{\cdot}\right|^n - 3\alpha\left|\frac{\alpha^2}{\cdot}\right|^n + 3\alpha^2\left|\frac{\alpha}{\cdot}\right|^n\right) + \frac{1}{2.3.4}\varphi^{iv}\alpha\left(\left|\frac{\alpha^4}{\cdot}\right|^n - 4\alpha\left|\frac{\alpha^3}{\cdot}\right|^n\right.$$

$$+6\alpha^2\left|\frac{\alpha^2}{.}\right|^n - 4\alpha^3\left|\frac{\alpha}{.}\right|^n\Big) + \&c. \ldots + \frac{1}{2.3\ldots n}\varphi^n\alpha\left(\left|\frac{\alpha^n}{.}\right|^n\right.$$

$$-n\alpha\left|\frac{\alpha^{n-1}}{.}\right|^n + \frac{n.n-1}{2}\left|\frac{\alpha^{n-2}}{.}\right|^n + \ldots . \pm n\alpha^{n-1}\left|\frac{\alpha}{.}\right|^n\Big).$$

Hence, also, supposing $n = r$.

$$84.\quad \mathfrak{C}^r = \left|\frac{\alpha^r}{.}\right|^r - r\alpha\left|\frac{\alpha^{r-1}}{.}\right|^r + \frac{r.r-1}{2}\alpha^2\left|\frac{\alpha^{r-2}}{.}\right|^r$$

$$-\frac{r.r-1.r-2}{2.3}\alpha^3\left|\frac{\alpha^{r-3}}{.}\right|^r + \ldots\ldots\ldots \pm r\alpha^{r-1}\left|\frac{\alpha}{.}\right|^r.$$

But when n is less than r, then

$$85.\quad o = \left|\frac{\alpha^r}{.}\right|^n - r\alpha\left|\frac{\alpha^{r-1}}{.}\right|^n + \frac{r.r-1}{2}\alpha^2\left|\frac{\alpha^{r-2}}{.}\right|^n$$

$$-\frac{r.r-1.r-2}{2.3}\alpha^3\left|\frac{\alpha^{r-3}}{.}\right|^n + \&c.$$

Or, since all the polynomial functions have the same index,

$$o = \left|\frac{\alpha^r}{1}\right|^n - r\alpha\left|\frac{\alpha^{r-1}}{1}\right|^n + \frac{r.r-1}{2}\alpha^2\left|\frac{\alpha^{r-2}}{1}\right|^n$$

$$-\frac{r.r-1.r-2}{2.3}\alpha^3\left|\frac{\alpha^{r-3}}{1}\right|^n + \&c.$$

For this series,

$$\left|\frac{\varphi\alpha}{1}\right|^0.z^0,\ \left|\frac{\varphi\alpha}{1}\right|'.z^1,\ \left|\frac{\varphi\alpha}{1.2}\right|''.z^2,\ \left|\frac{\varphi\alpha}{1.2.3}\right|'''.z^3,\ \left|\frac{\varphi\alpha}{2.3.4}\right|^{IV}.z^4,\ \&c.$$

is carried forward by a certain law, from which this rule for carrying it backward may be easily deduced, viz.:

Multiply any polynomial function by its index; then diminish that index by 1, and you get the preceding function. According to this rule, the four preceding terms would be,

$$0.1.2.3\left|\frac{\varphi\alpha}{1}\right|^{-IV}.z^{-4},\ 0.1.2\left|\frac{\varphi\alpha}{1}\right|^{-III}.z^{-3},\ 0.1.\left|\frac{\varphi\alpha}{1}\right|^{-II}.z^{-2},$$

$$0.\left|\frac{\varphi\alpha}{1}\right|^{-I}.z^{-1}.$$

But all these vanish, because 0 is their common multiple; so that every polynomial function, when accompanied with its proper multiplier, vanishes when its index becomes negative.

Lastly, the general term of log. $(\alpha - \mathfrak{C}z - \gamma z^2 - \delta z^3 - \text{\&c.})$, when $\alpha = 1$, may be thus developed;

$$86.\ \left|\frac{l\alpha}{.}\right|^n = \mathfrak{C}^n + n\mathfrak{C}^{n-2}\gamma$$

$$+n\mathfrak{C}^{n-3}\left|\frac{\gamma}{.}\right|'$$

$$+n\mathfrak{C}^{n-4}\left(\left|\frac{\gamma}{.}\right|'' + \frac{n-3}{2}\gamma^2\right)$$

$$+n\mathfrak{C}^{n-5}\left(\left|\frac{\gamma}{.}\right|''' + \frac{n-4}{2}\left|\frac{\gamma^2}{.}\right|'\right)$$

$$+n\mathfrak{C}^{n-6}\left(\left|\frac{\gamma}{.}\right|^{IV} + \frac{n-5}{2}\left|\frac{\gamma^2}{.}\right|'' + \frac{n-4.n-5}{2.3}\gamma^3\right)$$

$$+n\mathfrak{C}^{n-7}\left(\left|\frac{\gamma}{.}\right|^{v}+\frac{n-6}{2}\left|\frac{\gamma^2}{.}\right|'''+\frac{n-5\,.\,n-6}{2\,.\,3}\left|\frac{\gamma^3}{.}\right|'\right)$$

$$+n\mathfrak{C}^{n-8}\left(\left|\frac{\gamma}{.}\right|^{vi}+\frac{n-7}{2}\left|\frac{\gamma^2}{.}\right|^{iv}+\frac{n-6.n-7}{2.3}\left|\frac{\gamma^3}{.}\right|''+\frac{n-5..n-7}{2.3.4}\gamma^4\right)$$

$$+n\mathfrak{C}^{n-9}\left(\left|\frac{\gamma}{.}\right|^{vii}+\frac{n-8}{2}\left|\frac{\gamma^2}{.}\right|^{v}+\frac{n-7.n-8}{2.3}\left|\frac{\gamma^3}{.}\right|'''+\frac{n-6..n-8}{2.3.4}\left|\frac{\gamma^4}{.}\right|''\right)$$

&c. &c.

As may be easily deduced from 67., by giving each polynomial function its first developement in terms of γ, as in 60. 11–15.

SECTION XVII.

OF PRODUCTS COMPOSED OF EQUIDIFFERENT FACTORS.

It appears from Table II. (64.), that, if $\gamma = o$, $d = o$, &c. then

$$\left|\frac{\alpha^m}{1}\right|^n = m.\overline{m-1}.\overline{m-2}.\overline{m-3}\ldots\ldots\overline{m-n+1}.\alpha^{m-n}\mathfrak{C}^n.$$

Here, if $n=o$, $\left|\frac{\alpha^m}{.}\right|^o = \alpha^m$; if not, let n be any positive whole number, m any number whatever; and sup-

pose always, in the final result, $\alpha = 1$, and $\epsilon = 1$; then

87. THEOREM 1.

$$\left|\frac{\alpha^m}{1}\right|^n = m \,.\, m-1 \,.\, m-2 \,.\, m-3 \ldots\ldots \overline{m-n+1}.$$

COR. 1. $\left|\frac{\alpha^m}{1}\right|^m = 1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots m.$

COR. 2. $\left|\frac{\alpha^m}{1}\right|' = m$, $\left|\frac{\alpha^m}{1}\right|'' = m.m-1$, $\left|\frac{\alpha^m}{1}\right|''' = m.m-1.$ $m-2$, &c.

COR. 3. $\pm\left|\frac{\alpha^{-m}}{1}\right|^n = m.m+1.m+2 \ldots\ldots \overline{m+n-1}.$

$$\left(\pm \text{ if } n \begin{cases} \text{even} \\ \text{odd} \end{cases}\right).$$

88. THEOREM 2.

$$r^n \left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^n = m.m-r.m-2r \,.\, m-3r \ldots\ldots \overline{m-nr+r}.$$

Where m is the first factor, r the common difference, and n the number of factors.

COR. 1. $r\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|' = m$, $r^2\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|'' = m \,.\, m-r$,

$$r^3\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|''' = m \,.\, m-r \,.\, m-2r, \text{ &c.}$$

Cor. 2. $\pm r^n \left|\begin{matrix}\alpha^{-\frac{m}{r}} \\ 1\end{matrix}\right|^n = m.m+r.m+2r.m+3r.... \overline{m+nr-r}.$

89. THEOREM 3.

$$\pm r^n \left|\begin{matrix}\alpha^{-\frac{m}{r}} \\ 1\end{matrix}\right|^n = r^n \left|\begin{matrix}\alpha^{\frac{m}{r}+n-1} \\ 1\end{matrix}\right|^n = r^n \left|\begin{matrix}\alpha^{\frac{m+nr-r}{r}} \\ 1\end{matrix}\right|^n.$$

For this last $= \overline{m+nr-r}.\overline{m+nr-2r}.\overline{m+nr-3r}.....m$, (88.), which is also the developement of the first, (88. Cor. 2.)

Cor. $\pm \left|\begin{matrix}\alpha^{-\frac{m}{r}} \\ 1\end{matrix}\right|^n = \left|\begin{matrix}\alpha^{\frac{m}{r}+n-1} \\ 1\end{matrix}\right|^n$. (89.) In like manner,

$$\pm \left|\begin{matrix}\alpha^{-m-1} \\ 1\end{matrix}\right|^n = \left|\begin{matrix}\alpha^{m+n} \\ 1\end{matrix}\right|^n. \quad \&c.$$

90. THEOREM 4.

$$\left|\begin{matrix}\alpha^{\frac{m}{r}} \\ 1\end{matrix}\right|^{n+t} = \left|\begin{matrix}\alpha^{\frac{m}{r}} \\ 1\end{matrix}\right|^n \times \left|\begin{matrix}\alpha^{\frac{m}{r}-n} \\ 1\end{matrix}\right|^t = \left|\begin{matrix}\alpha^{\frac{m}{r}} \\ 1\end{matrix}\right|^t \times \left|\begin{matrix}\alpha^{\frac{m}{r}-t} \\ 1\end{matrix}\right|^n.$$

For $r^{n+t} \left|\begin{matrix}\alpha^{\frac{m}{r}} \\ 1\end{matrix}\right|^{n+t} = m.m-r.m-2r.....\overline{m-(n+t)r+r}$,

(88.), which may be resolved into these two products,

$$(m.m-r.m-2r.....m-nr+r)$$
$$\times (\overline{m-nr}, \overline{m-nr-r}....\overline{m-(n+t)r+r});$$

of which the former $= r^n \left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^n$, and the latter $= r^t \left|\frac{\alpha^{\frac{m-nr}{r}}}{1}\right|^t$. Hence the proposition is manifest.

Cor. 1. Hence,

$$\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-n} = \frac{1}{m+r \,.\, m+2r \,.\, m+3r \,.\,.\,.\,.\,.\,.\, m+nr}.$$

For, if $n + t = o$, or $t = -n$, then

$\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{o} = \left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-n} \times \left|\frac{\alpha^{\frac{m+nr}{r}}}{1}\right|^{n}$; but $\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{o} = \alpha^{\frac{m}{r}} = 1$; therefore,

$$\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-n} = \frac{1}{\left|\frac{\alpha^{\frac{m+nr}{r}}}{1}\right|^{n}} = \frac{1}{m+nr \,.\, m+nr-r \,.\,.\,.\,.\,.\, m+r}.$$

Cor. 2. $\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-1} = \frac{1}{m+r}$, $\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-11} = \frac{1}{m+r \,.\, m+2r}$, &c.

Cor. 3. $\pm r^n \left|\frac{\alpha^{-\frac{m}{r}}}{1}\right| \times \left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-n} = \frac{m}{m+r}$.

Cor. 4. $\left|\frac{\alpha^{2n}}{1}\right|^{2n} = \left|\frac{\alpha^{2n}}{1}\right|^{n} \times \left|\frac{\alpha^{n}}{1}\right|^{n}$, by putting $m = 2n$, $r = 1$, and $t = n$.

Thus, also, $\left|\frac{\alpha^{3n}}{1}\right|^{3n} = \left|\frac{\alpha^{3n}}{1}\right|^{n} \cdot \left|\frac{\alpha^{2n}}{1}\right|^{n} \cdot \left|\frac{\alpha^{n}}{1}\right|^{n}$, &c.

91. THEOREM 5.

$$\left|\frac{\alpha^{2n}|^{2n}}{1}\right| = 2^n \left|\frac{\alpha^n|^n}{1}\right| \cdot 2^n \left|\frac{\alpha^{\frac{2n-1}{2}}|^n}{1}\right|.$$

For $\left|\frac{\alpha^{2n}|^{2n}}{1}\right| = 2n \,.\, 2n-1 \,.\, 2n-2 \,.\, 2n-3 \ldots\ldots 1,$

may be resolved into these two products, viz.

$$(2n \,.\, 2n-2 \,.\, 2n-4 \ldots\ldots 2) = 2^n \left|\frac{\alpha^{\frac{2n}{2}}|^n}{1}\right|,$$

and $$(2n-1 \,.\, 2n-3 \,.\, 2n-5 \ldots\ldots 1) = 2^n \left|\frac{\alpha^{\frac{2n-1}{2}}|^n}{1}\right| \quad (88.)$$

Cor. 1. Hence $\left|\frac{\alpha^{2n}|^n}{1}\right| = 2^n \,.\, 2^n \,.\, \left|\frac{\alpha^{\frac{2n-1}{2}}|^n}{1}\right|$. (90. Cor. 4.)

Cor. 2. In like manner,

$$\left|\frac{\alpha^{3n}|^{3n}}{1}\right| = 3^n \left|\frac{\alpha^n|^n}{1}\right| \cdot 3^n \left|\frac{\alpha^{\frac{3n-1}{3}}|^n}{1}\right| \cdot 3^n \left|\frac{\alpha^{\frac{3n-2}{3}}|^n}{1}\right|.$$

Cor. 3. Hence $\left|\frac{\alpha^{3n}|^n}{1}\right| \cdot \left|\frac{\alpha^{2n}|^n}{1}\right| = 3^{2n} \left|\frac{\alpha^{\frac{3n-1}{3}}|^n}{1}\right| \cdot \left|\frac{\alpha^{\frac{3n-2}{3}}|^n}{1}\right|.$

We shall next consider the properties of these polynomial functions, when they are accompanied with their proper divisors.

92. THEOREM 6.

$$\pm \left|\begin{matrix} \alpha^{\frac{-m}{r}} \\ \cdot \end{matrix}\right|^n = \left|\begin{matrix} \alpha^{\frac{m}{r}+n-1} \\ \cdot \end{matrix}\right|^n = \left|\begin{matrix} \alpha^{\frac{m+nr-r}{r}} \\ \cdot \end{matrix}\right|^n. \quad (89.\ \text{Cor.})$$

also $\pm \left|\begin{matrix} \alpha^{-m-1} \\ \cdot \end{matrix}\right|^n = \left|\begin{matrix} \alpha^{m+n} \\ \cdot \end{matrix}\right|^n$, &c.

93. THEOREM 7.

$$\left|\begin{matrix} \alpha^m \\ \cdot \end{matrix}\right|^n = \left|\begin{matrix} \alpha^n \\ \cdot \end{matrix}\right|^{m-n}.$$

For $\dfrac{m.m-1.m-2.....m-n+1.}{1.2.3.....n} = \dfrac{m.m-1.m-2.....,n+1}{1.2.3.....m-n}$.

Thus, if, for n, we substitute any determinate number, as 4, we have,

$$\frac{m.m-1.m-2.m-3}{1.2.3.4} = \frac{m.m-1.m-2.m-3}{1.2.3.4}\,\Big|\,\frac{m-4.m-5....6.5}{5.6....m-5.m-4}.$$

COR. From the last two theorems, it follows that

$$\pm \left|\begin{matrix} \alpha^{-m-1} \\ \cdot \end{matrix}\right|^n = \left|\begin{matrix} \alpha^{m+n} \\ \cdot \end{matrix}\right|^n = \left|\begin{matrix} \alpha^{m+n} \\ \cdot \end{matrix}\right|^m.$$

94. THEOREM 8.

$$\left|\begin{matrix} \alpha^{m+n} \\ \cdot \end{matrix}\right|^n = \alpha^m \left|\begin{matrix} \alpha' \\ \cdot \end{matrix}\right|^n + \left|\begin{matrix} \alpha^m \\ \cdot \end{matrix}\right|' \cdot \left|\begin{matrix} \alpha' \\ \cdot \end{matrix}\right|^{n-1} + \left|\begin{matrix} \alpha^m \\ \cdot \end{matrix}\right|'' \cdot \left|\begin{matrix} \alpha' \\ \cdot \end{matrix}\right|^{n-2} + \text{&c.}$$

This is a particular case of the theorem in Article 72.

Cor. 1. Suppose $s = n$; then because $\left|\frac{\alpha^s}{.}\right|^{n-1} = \left|\frac{\alpha^n}{.}\right|'$, &c. (93.)

$$\left|\frac{\alpha^{m+n}}{.}\right|^n = \alpha^m \left|\frac{\alpha^n}{.}\right|^n + \left|\frac{\alpha^m}{.}\right|' \cdot \left|\frac{\alpha^n}{.}\right|' + \left|\frac{\alpha^m}{.}\right|'' \cdot \left|\frac{\alpha^n}{.}\right|'' + \ldots + \left|\frac{\alpha^m}{.}\right|^n \cdot \left|\frac{\alpha^n}{.}\right|^n;$$

that is, $$\frac{m+1 \,.\, m+2 \,.\, m+3 \ldots . m+n}{1 \,.\, 2 \,.\, 3 \ldots . n} = 1 + mn + \frac{m \,.\, m-1}{2} \times \frac{n \,.\, n-1}{2} + \frac{m \,.\, m-1 \,.\, m-2}{2 \,.\, 3} \times \frac{n \,.\, n-1 \,.\, n-2}{2 \,.\, 3}$$

+ &c. = sum of the products of the corresponding co-efficients in the powers, m, n, of two binomials.

Cor. 2. Suppose, next, $m \pm s = n$; then we have,

$$\left|\frac{\alpha^{2n}}{.}\right|^n = \alpha^n + \left(\left|\frac{\alpha^n}{.}\right|'\right)^2 + \left(\left|\frac{\alpha^n}{.}\right|''\right)^2 + \left(\left|\frac{\alpha^n}{.}\right|'''\right)^2 + \ldots + \left(\left|\frac{\alpha^n}{.}\right|^n\right)^2;$$

that is, $$\frac{n+1 \,.\, n+2 \,.\, n+3 \ldots . 2n}{1 \,.\, 2 \,.\, 3 \ldots . n} = 1 + n^2 + \left(\frac{n \,.\, n-1}{2}\right)^2 + \left(\frac{n \,.\, n-1 \,.\, n-2}{2 \,.\, 3}\right)^2 + \text{&c.} = \text{sum}$$

of the squares of the co-efficients in the n power o a binomial.

Cor. 3. Because $\left|\frac{\alpha^{2n}}{.}\right|^n = 2^n \,.\, 2^n \left|\frac{\alpha^{\frac{2n-1}{2}}}{.}\right|^n$, the said sum is also equal $\frac{2n-1 \,.\, 2n-3 \,.\, 2n-5 \,.\, 2n-7 \ldots\ldots 1}{1 \,.\, 2 \,.\, 3 \ldots\ldots n} \,.\, 2^n$.

Hence $2^n = \frac{n+1 \,.\, n+2 \,.\, n+3 \ldots 2n}{1 \,.\, 3 \,.\, 5 \,.\, 7 \ldots . 2n-1}$,

Cor. 4. Again, if, for m and s, we take $-m$ and -1, we have,

$$\left|\frac{\alpha^{-m-1}}{\cdot}\right|^n = \alpha^{-m}\left|\frac{\alpha^{-1}}{\cdot}\right|^n + \left|\frac{\alpha^{-m}}{\cdot}\right|' \cdot \left|\frac{\alpha^{-1}}{\cdot}\right|^{n-1} + \left|\frac{\alpha^{-m}}{\cdot}\right|'' \cdot \left|\frac{\alpha^{-1}}{\cdot}\right|^{n-2} + \ldots\ldots + \left|\frac{\alpha^{-m}}{\cdot}\right|^n \cdot \alpha^{-1}.$$

Now, the polynomial functions of α^{-m} are negative or affirmative, according as the index is odd or even; those of α^{-1} are each $= 1$, and negative or affirmative by the same rule. Hence

$$\left|\frac{\alpha^{-m-1}}{\cdot}\right|^n = \alpha^{-m} + \left|\frac{\alpha^{-m}}{\cdot}\right|' + \left|\frac{\alpha^{-m}}{\cdot}\right|'' + \left|\frac{\alpha^{-m}}{\cdot}\right|''' + \ldots\ldots + \left|\frac{\alpha^{-m}}{\cdot}\right|^n;$$

that is, $$\frac{m+1 \,.\, m+2 \ldots\ldots m+n}{1 \,.\, 2 \,.\, 3 \ldots n} = 1 + m + \frac{m \,.\, m+1}{2} + \frac{m \,.\, m+1 \,.\, m+2}{2 \,.\, 3} + \text{\&c. (to } n+1 \text{ terms).}$$

Cor. 5. In the same manner, we find

$$\left|\frac{\alpha^{m-1}}{\cdot}\right|^n = \alpha^m - \left|\frac{\alpha^m}{\cdot}\right|' + \left|\frac{\alpha^m}{\cdot}\right|'' - \left|\frac{\alpha^m}{\cdot}\right|''' + \ldots\ldots \pm \left|\frac{\alpha^m}{\cdot}\right|^n;$$

that is, $$\frac{m-1 \,.\, m-2 \ldots\ldots m-n}{1 \,.\, 2 \,.\, 3 \ldots n} = 1 - m + \frac{m \,.\, m-1}{2} - \frac{m \,.\, m-1 \,.\, m-2}{2 \,.\, 3} + \text{\&c. (to } n+1 \text{ terms).}$$

Hence, (when $m = n$,) $$o = 1 - m + \frac{m \,.\, m-1}{2} - \frac{m \,.\, m-1 \,.\, m-2}{2 \,.\, 3} + \ldots\ldots \pm 1.$$

95. THEOREM 9.

$$\left|\frac{\alpha^{m+s}}{.}\right|^{m+n} = \alpha^{m}\left|\frac{\alpha^{s}}{.}\right|^{n} + \left|\frac{\alpha^{m}}{.}\right|'.\left|\frac{\alpha^{s}}{.}\right|^{n+1} + \left|\frac{\alpha^{m}}{.}\right|''.\left|\frac{\alpha^{s}}{.}\right|^{n+2} + \ldots + \left|\frac{\alpha^{m}}{.}\right|^{n}.\alpha^{s}.$$

For $\left|\frac{\alpha^{m+s}}{.}\right|^{m+n} = \left|\frac{\alpha^{m+s}}{.}\right|^{s-n}$ (93.), which being developed by the preceding theorem, and $\left|\frac{\alpha^{s}}{.}\right|^{n}.\left|\frac{\alpha^{s}}{.}\right|^{n+1}$, &c. substituted in place of $\left|\frac{\alpha^{s}}{.}\right|^{s-n}.\left|\frac{\alpha^{s}}{.}\right|^{s-n-1}$, &c. we get Theorem 9.

96. THEOREM 10.

$$\left|\frac{\alpha^{\frac{s+m+nr}{r}}}{1}\right|^{n} \times \left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-n} = 1 + n\left|\frac{\alpha_{s}}{.}\right|'.\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-I} + \frac{n.n-1}{2}\left|\frac{\alpha_{s}}{1}\right|''.\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-II}$$
$$+ \frac{n.n-1.n-2}{2.3}\left|\frac{\alpha^{s}}{1}\right|'''.\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-III} + \&c.$$

$$\text{For } \left|\frac{\alpha^{\frac{s+m+nr}{r}}}{1}\right|^{n} = \alpha^{\frac{s}{r}}\left|\frac{\alpha^{\frac{m+nr}{r}}}{1}\right|^{n} + n\left|\frac{\alpha^{\frac{s}{r}}}{1}\right|'.\left|\frac{\alpha^{\frac{m+nr}{r}}}{1}\right|^{n-I}$$
$$+ \frac{n.n-1}{2}\left|\frac{\alpha^{\frac{s}{r}}}{1}\right|''.\left|\frac{\alpha^{\frac{m+nr}{r}}}{1}\right|^{n-II} + \&c.$$

which, being multiplied by $\left|\frac{\alpha^{\frac{m}{r}}}{1}\right|^{-n}$, may be changed by Theorem 4. to the expression above.

SECTION XVIII.

OF COMPOUND POLYNOMES.

97. LET it be required to develope the compound polynome,

$$a + b \cdot \pi x^p + c \cdot \pi_{,} x^{p+q} + d \cdot \pi_{,,} x^{p+2q} + e \cdot \pi_{,,,} x^{p+3q} + \&c.$$

(where π, $\pi_{,}$, $\pi_{,,}$, &c. denote so many different and independent polynomes of this form, $\alpha + \xi x^q + \gamma x^{2q} + \delta x^{3q} +$ &c.) into the simple polynome,

$$A + Bx^p + Cx^{p+q} + Dx^{p+2q} + Ex^{p+3q} + \&c.$$

Because $\pi = \varphi(\alpha + \xi x^q + \gamma x^{2q} + \delta x^{3q} + \&c.)$

$$= \varphi\alpha + \left|\frac{\varphi\alpha}{\cdot}\right|' \cdot x^q + \left|\frac{\varphi\alpha}{\cdot}\right|'' \cdot x^{2q} + \left|\frac{\varphi\alpha}{\cdot}\right|''' \cdot x^{3q} + \&c.$$

$$\pi_{,} = F\alpha_{,} + \left|\frac{F\alpha_{,}}{\cdot}\right|' \cdot x^q + \left|\frac{F\alpha_{,}}{\cdot}\right|'' \cdot x^{2q} + \left|\frac{F\alpha_{,}}{\cdot}\right|''' \cdot x^{3q} + \&c.$$

$$\pi_{,,} = f\alpha_{,,} + \left|\frac{f\alpha_{,,}}{\cdot}\right|' \cdot x^q + \left|\frac{f\alpha_{,,}}{\cdot}\right|'' \cdot x^{2q} + \&c.$$

&c. &c.

By substituting these values in the given series, it becomes

$$a+\varphi\alpha.bx+\left|\frac{\varphi\alpha}{\cdot}\right|'.bx^{p+q}+\left|\frac{\varphi\alpha}{\cdot}\right|''.bx^{p+2q}+\left|\frac{\varphi\alpha}{\cdot}\right|'''.bx^{p+3q}+\&c.$$
$$+F\alpha_{/}.cx^{p+q}+\left|\frac{F\alpha_{/}}{\cdot}\right|'.cx^{p+2q}+\left|\frac{F\alpha_{/}}{\cdot}\right|''.cx^{p+3q}+\&c.$$
$$+f\alpha_{//}.dx^{p+2q}+\left|\frac{f\alpha_{//}}{\cdot}\right|'.dx^{p+3q}+\&c.$$
$$\&c. \qquad\qquad \&c.$$

Hence $A = a.$

$$B = \varphi\alpha . b.$$
$$C = \left|\frac{\varphi\alpha}{\cdot}\right|'.b + F\alpha_{/}.c.$$
$$D = \left|\frac{\varphi\alpha}{\cdot}\right|''.b + \left|\frac{F\alpha_{/}}{\cdot}\right|'.c + f\alpha_{//}.d.$$
$$\&c. \qquad\qquad \&c.$$

The law of continuation is obvious. It is evident, too, that, by means of the same general equations, the converse of the problem may be resolved. For the co-efficients A, B, C, &c. and the several functions, $\varphi\alpha$, $F\alpha_{/}$, $f\alpha_{//}$, &c. with their polynomial derivatives being given, the co-efficients a, b, c, &c. of the required series may be successively obtained from them.

Thus,

$a = \mathrm{A}.$

$b = (\varphi\alpha)^{-1} . \mathrm{B}.$

$$c = (\mathrm{F}\alpha_{,})^{-1} . \left(\mathrm{C} - (\varphi\alpha)^{-1} . \left|\frac{\varphi\alpha}{.}\right|' . \mathrm{B}\right).$$

$$d = (f\alpha_{,,})^{-1} . \left(\mathrm{D} - (\mathrm{F}\alpha_{,})^{-1} . \left|\frac{\varphi\alpha}{.}\right|' \mathrm{C} + (\varphi\alpha . \mathrm{F}\alpha_{,})^{-1} . \left|\frac{\varphi\alpha}{.}\right|' . \left|\frac{\mathrm{F}\alpha_{,}}{.}\right|' . \mathrm{B} - (\varphi\alpha)^{-1} . \left|\frac{\varphi\alpha}{.}\right|'' . \mathrm{B}\right).$$

&c. &c.

We shall now consider some of the most useful cases of this very general problem.

CASE I.

98. Supposing $s = \alpha + \xi x^{q} + \gamma x^{2q} + \delta x^{3q} +$ &c. let one of these two equal series be given, to find the other, viz.

$$a + bs^{m}x^{p} + cs^{2m}x^{p+q} + ds^{3m}x^{p+2q} + es^{4m}x^{p+3q} + \text{&c.}$$
$$= \mathrm{A} + \mathrm{B}x^{p} + \mathrm{C}x^{p+q} + \mathrm{D}x^{p+2q} + \mathrm{E}x^{p+3q} + \text{&c.}$$

From the general solution, we have,

$\mathrm{A} = a.$

$\mathrm{B} = \alpha^{m}b.$

$$C = \left|\frac{\alpha^{m}}{\cdot}\right|' . b + \alpha^{2m} c.$$

$$D = \left|\frac{\alpha^{m}}{\cdot}\right|'' . b + \left|\frac{\alpha^{2m}}{\cdot}\right|' . c + \alpha^{3m} d.$$

. . . . . .

$$A_n = \left|\frac{\alpha^{m}}{\cdot}\right|^{n-1} . b + \left|\frac{\alpha^{2m}}{\cdot}\right|^{n-2} . c + \left|\frac{\alpha^{3m}}{\cdot}\right|^{n-3} . d + \ldots + \alpha^{nm} . a_n.$$

Also,

$$a = A.$$

$$b = \alpha^{-m} . B.$$

$$c = \alpha^{-2m} . C - \alpha^{-3m} . \left|\frac{\alpha^{m}}{\cdot}\right|' . B.$$

$$d = \alpha^{-3m} . D - \alpha^{-5m} . \left|\frac{\alpha^{2m}}{\cdot}\right|' . C - \alpha^{-4m} . \left|\frac{\alpha^{m}}{\cdot}\right|'' + \alpha^{-6m} . \left|\frac{\alpha^{m}}{\cdot}\right|' . \left|\frac{\alpha^{2m}}{\cdot}\right|' \Bigg\} B.$$

&c.

And, by means of the Theorems, (76–80,) we get these more convenient expressions for the co-efficients, c, d, e, &c.

$$b = \alpha^{-m} B.$$

$$c = \alpha^{-2m} C + \tfrac{1}{2} \left|\frac{\alpha^{-2m}}{\cdot}\right|' . B.$$

$$d = \alpha^{-3m} D + \tfrac{2}{3} \left|\frac{\alpha^{-3m}}{\cdot}\right|' . C + \tfrac{1}{3} \left|\frac{\alpha^{-3m}}{\cdot}\right|'' . B.$$

$$e = \alpha^{-4m}\mathrm{E} + \tfrac{3}{4}\left|\frac{\alpha^{-4m}}{\cdot}\right|'.\mathrm{D} + \tfrac{2}{4}\left|\frac{\alpha^{-4m}}{\cdot}\right|''.\mathrm{C} + \tfrac{1}{4}\left|\frac{\alpha^{-4m}}{\cdot}\right|'''.\mathrm{B}.$$

. . . . . .

$$a_n = \alpha^{-nm}\mathrm{A}_n + \frac{n-1}{n}\left|\frac{\alpha^{-nm}}{\cdot}\right|'\mathrm{A}_{n-1} + \frac{n-2}{n}\left|\frac{\alpha^{-nm}}{\cdot}\right|''\mathrm{A}_{n-2}$$

$$+ \ldots\ldots\ldots\ldots\ldots \frac{2}{n}\left|\frac{\alpha^{-nm}}{\cdot}\right|^{n-2}.\mathrm{C} + \frac{1}{n}\left|\frac{\alpha^{-nm}}{\cdot}\right|^{n-1}.\mathrm{B}.$$

Which, again, may be written more simply and more concisely, thus,

$$b = \alpha^{-m}\mathrm{A}'.$$

$$c = \tfrac{1}{2}\left|\frac{\alpha^{-2m}.\mathrm{A}'}{\cdot}\right|'.$$

$$d = \tfrac{1}{3}\left|\frac{\alpha^{-3m}.\mathrm{A}'}{\cdot}\right|''.$$

$$e = \tfrac{1}{4}\left|\frac{\alpha^{-4m}.\mathrm{A}'}{\cdot}\right|'''.$$

. . . . . .

$$a_n = \frac{1}{n}\left|\frac{\alpha^{-nm}.\mathrm{A}'}{\cdot}\right|^{n-1}.$$

For the former expression for a_n will easily appear (by Theorem, Article 72.) to be the development of the latter; the n^{th} polynomial derivative of A (or $\overline{n-1}$ of A′) being na_n.

Hence this Theorem:

$$99.\ A + Bx^p + Cx^{p+q} + Dx^{p+2q} + \&c.$$
$$= A + \tfrac{1}{1}\left|\frac{\alpha^{-m}.A'}{.}\right|^{0}.s^m x^p + \tfrac{1}{2}\left|\frac{\alpha^{-2m}.A''}{.}\right|'.s^{2m}.x^{p+q}$$
$$+ \tfrac{1}{3}\left|\frac{\alpha^{-3m}.A'}{.}\right|''.s^{3m}x^{p+2q} + \ldots\ldots\ldots$$
$$+ \frac{1}{n}\left|\frac{\alpha^{-nm}.A'^{n-1}}{.}\right|.s^{nm}x^{p+nq}.$$

Where $s = \alpha + \xi x^q + \gamma x^{2q} + \delta x^{3q} + \&c.$

CASE II.

$$100.\ a + bs^m x^p + cs^{2m}x^{p+q} + ds^{3m}x^{p+2q} + es^{4m}x^{p+3q} + \&c.$$
$$= A + Bx^p + Cx^{p+q} + Dx^{p+2q} + Ex^{p+3q} + \&c.$$
$$s = \alpha + \xi x^q + \gamma x^{2q} + \delta x^{3q} + \&c.\ \text{as before.}$$

Here

$A = a.$

$B = \alpha^m b.$

$C = \left|\frac{\alpha^m}{.}\right|' b + \alpha^{2m}c.$

$D = \left|\frac{\alpha^m}{.}\right|'' b + \left|\frac{\alpha^{2m}}{.}\right|' c + \alpha^{3m}d.$

. . . . . .

$A_n = \left|\frac{\alpha^m}{.}\right|^{n-1} b + \left|\frac{\alpha^{2m}}{.}\right|^{n-2} c + \left|\frac{\alpha^{3m}}{.}\right|^{n-3} d + \ldots + \alpha^{\overline{-n-1}.m}.a_n.$

Hence, as in the preceding case, we get these values of b, c, d, &c.

$$b = \alpha^{-m} \mathrm{B}.$$

$$c = \alpha^{-2m} \mathrm{C} + \tfrac{1}{2} \left| \frac{\alpha^{-2m}}{\cdot} \right|' \mathrm{B}.$$

$$d = \alpha^{-3m} \mathrm{D} + \tfrac{2}{3} \left| \frac{\alpha^{-2m}}{\cdot} \right|' \mathrm{C} + \tfrac{1}{3} \left| \frac{\alpha^{-2m}}{\cdot} \right|'' \mathrm{B}.$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$a_n = \alpha^{-\overline{2n-1}.m} \mathrm{A}_n + \frac{2n-3}{2n-1} \left| \frac{\alpha^{-\overline{2n-1}.m}}{\cdot} \right|' . \mathrm{A}_{n-1}$$

$$+ \frac{2n-5}{2n-1} \left| \frac{\alpha^{-\overline{2n-1}.m}}{\cdot} \right|'' . \mathrm{A}_{n-2} + \cdots\cdots\cdots$$

$$\frac{1}{2n-1} \left| \frac{\alpha^{-\overline{2n-1}.m}}{\cdot} \right|^{n-1} . \mathrm{B}.$$

CASE III.

$$a + bs^m . x^p + cs^{m+\partial} . x^{p+q} + ds^{m+2\partial} . x^{p+2q} + es^{m+3\partial} . x^{p+3q} + \&c.$$
$$= \mathrm{A} + \mathrm{B}x^p + \mathrm{C}x^{p+q} + \mathrm{D}x^{p+2q} + \&c.$$
$$s = \text{as above.}$$

This Case includes the two former; and its more general values of b, c, d, &c. may easily be deduced, independently of them, from the general solution, by means of Theorem 6. (81.). Thus,

$$b = \alpha^{-m} \cdot B.$$

$$c = \alpha^{-m-\partial} \cdot C + \frac{m}{m+\partial} \left| \frac{\alpha^{-m-\partial}}{\cdot} \right|' \cdot B.$$

$$d = \alpha^{-m-2\partial} \cdot D + \frac{m+\partial}{m+2\partial} \left| \frac{\alpha^{-m-2\partial}}{\cdot} \right|' \cdot C + \frac{m}{m+2\partial} \left| \frac{\alpha^{-m-2\partial}}{\cdot} \right|'' \cdot B.$$

$$e = \alpha^{-m-3\partial} \cdot E + \frac{m+2\partial}{m+3\partial} \left| \frac{\alpha^{-m-3\partial}}{\cdot} \right|' \cdot D + \frac{m+\partial}{m+3\partial} \left| \frac{\alpha^{-m-3\partial}}{\cdot} \right|'' \cdot C + \frac{m}{m+3\partial} \left| \frac{\alpha^{-m-3\partial}}{\cdot} \right|''' \cdot B.$$

&c. &c.

SECTION XIX.

OF THE REVERSION OF SERIES.

THE application of the preceding Cases to the reversion of series is simple and elegant.

PROBLEM 1.

101. Given $z = \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 +$ &c. to find $x = bz + cz^2 + dz^3 + ez^4 +$ &c.

By substituting for z, in the second equation, its value in the first, (putting $s = \alpha + \beta x + \gamma x^2 + \delta x^3 + \&c.$) we have,

$$x = bsx + cs^2x^2 + ds^3x^3 + es^4x^4 + \&c.$$

But this compound series (by Case I. 98.) being equivalent to the simple one, $A + Bx + Cx^2 + Dx^3 + \&c.$ we have also,

$$x = A + Bx + Cx^2 + \&c.$$

Whence $A = o$, $B = A' = 1$, $C = o$, $D = o$, &c.; and since $m = 1$, we get the following simple expressions for the co-efficients, b, c, d, &c. viz.

$$b = \alpha^{-1}.$$

$$c = \tfrac{1}{2}\left|\frac{\alpha^{-2}}{\cdot}\right|'.$$

$$d = \tfrac{1}{3}\left|\frac{\alpha^{-3}}{\cdot}\right|''.$$

. . . . . .

$$a_n = \frac{1}{n}\left|\frac{\alpha^{-n}}{\cdot}\right|^{n-1}.$$

Hence

102. $$x = \alpha^{-1}z + \tfrac{1}{2}\left|\frac{\alpha^{-2}}{\cdot}\right|' \cdot z^2 + \tfrac{1}{3}\left|\frac{\alpha^{-3}}{\cdot}\right|'' \cdot z^3 + \tfrac{1}{4}\left|\frac{\alpha^{-4}}{\cdot}\right|''' + \ldots + \frac{1}{n}\left|\frac{\alpha^{-n}}{\cdot}\right|^{n-1}.$$

And these polynomial derivatives are easily developed by means of Table II. (64.). Thus,

$$x = \alpha^{-1}z - \alpha^{-3}\mathfrak{C}z^2 - (\alpha^{-4}\gamma - \tfrac{4}{2}\alpha^{-5}\mathfrak{C}^2)z^3 - (\alpha^{-5}\delta - \tfrac{5}{2}\alpha^{-6}.2\mathfrak{C}\gamma + \frac{5.6}{2.3}\alpha^{-7}\mathfrak{C}^3)z^4 - (\alpha^{-6}\varepsilon - \tfrac{6}{2}\alpha^{-7}.2\mathfrak{C}\delta + \gamma^2 + \frac{6.7}{2.3}\alpha^{-8}.3\mathfrak{C}^2\gamma - \frac{6.7.8}{2.3.4}\alpha^{-9}.\mathfrak{C}^4)z^5 - \&c.$$

Which expression, if $\alpha^{-1} = 1$, becomes

$$x = z - \mathfrak{C}z^2 - (\gamma - 2\mathfrak{C}^2)z^3 - (\delta - 5\mathfrak{C}\gamma + 5\mathfrak{C}^3)z^4 - (\varepsilon - 6\mathfrak{C}\delta - 3\gamma^2 + 21\mathfrak{C}^2\gamma - 14\mathfrak{C}^4)z^5 - \Big(\zeta - \tfrac{7}{2}(2\mathfrak{C}\varepsilon + 2\gamma\delta) + \frac{7.8}{2.3}(3\mathfrak{C}^2\delta + 3\mathfrak{C}^2\gamma) - \frac{7.8.9}{2.3.4}.4\mathfrak{C}^3\gamma + \frac{7.8.9.10}{2.3.4.5}\mathfrak{C}^5\Big)z^6 - \Big(\eta - \tfrac{8}{2}(2\mathfrak{C}\zeta + 2\gamma\varepsilon + \delta^2) + \frac{8.9}{2.3}(3\mathfrak{C}^2\varepsilon + 6\mathfrak{C}\gamma\delta + \gamma^3) - \frac{8.9.10}{2.3.4}(4\mathfrak{C}^3\delta + 6\mathfrak{C}^2\gamma^2) + \frac{8.9.10.11}{2.3.4.5}.5\mathfrak{C}^4\gamma - \frac{8.9.10.11.12}{2\ldots.6}\mathfrak{C}^6\Big)z^7 - \&c.$$

PROBLEM II.

103. Given $z = \alpha x + \mathfrak{C}x^2 + \gamma x^3 + \delta x^4 + \&c.$

to find $x = bz + cz^2 + dz^3 + ez^4 + \&c.$

Here $s = \alpha + \mathfrak{C}x + \gamma x^2 + \delta x^3 + \&c.$ And

$x = bsx + cs^2x^2 + ds^3x^3 + \&c. = A + Bv + Cv^2 + Dv^3 +, \&c.$

Hence $A = o$, $B = A' = 1$, $C = o$, &c. and $m = 1$, $p = 1$, $q = 2$. So that all the terms which compose the values of b, c, d, &c. in Case II, except the last, vanish; we have therefore a similar expression for x, viz.:

[illegible]

[illegible] &c.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] &c.

[illegible] &c.

[illegible]

[illegible] &c. $= A + Bx^{\theta}$

[illegible]

[illegible] (see Case III.) $m = 1$,

[illegible] Therefore $z^{\theta} = x^{-\theta} + \frac{1}{1+\theta} \frac{x^{-1-\theta}}{\cdot} \Big|' \cdot x^{\theta+1}$

$+ \frac{1}{1+2\theta} \frac{x^{-1-2\theta}}{\cdot} \Big|'' \cdot x^{\theta+2} + \frac{1}{1+3\theta} \frac{x^{-1-3\theta}}{\cdot} \Big|''' \cdot x^{\theta+3} +$ &c.

PROBLEM IV.

105. Given $z=x(\alpha+\mathfrak{C}x^q+\gamma x^{2q}+\delta x^{3q}+\&c.)^p$ to find x^r.

Here $z^r = x^r(\alpha + \mathfrak{C}x^q + \gamma x^{2q} + \&c.)^{pr}$,

or, $z^r = \alpha^{pr}.x^r + \left|\frac{\alpha^{pr}}{.}\right|'.x^{r+q} + \left|\frac{\alpha^{pr}}{.}\right|''.x^{r+2q} + \&c.$

Hence, as in the preceding problem, we get

$$x^r=\alpha^{-pr}.z^r+\frac{r}{r+q}\left|\frac{\alpha^{-pr-pq}}{.}\right|'.z^{r+q}+\frac{r}{r+2q}\left|\frac{\alpha^{-pr-2pq}}{.}\right|''.z^{r+2q}+\&c.$$

Thus, if $z=\alpha x+\mathfrak{C}x^2+\gamma x^3+\delta x^4+\&c.$, then $p=1$, $q=1$.

106. And $x^r=\alpha^{-r}z^r+\frac{r}{r+1}\left|\frac{\alpha^{-r-1}}{.}\right|'.z^{r+1}+\frac{r}{r+2}\left|\frac{\alpha^{-r-2}}{.}\right|''.z^{r+2}$

$+\frac{r}{r+3}\left|\frac{\alpha^{-r-3}}{.}\right|'''.z^{r+3}+\&c.=\alpha^{-r}z^r-r\alpha^{-r-2}.\mathfrak{C}z^{r+1}-(r\alpha^{-r-3}\gamma$

$-\frac{r.r+3}{1.2}\alpha^{-r-4}.\mathfrak{C}^2).z^{r+2}-(r\alpha^{-r-4}\delta-\frac{r.r+4}{1.2}\alpha^{-r-5}.2\mathfrak{C}\gamma$

$+\frac{r.r+4.r+5}{1.2.3}\alpha^{-r-6}.\mathfrak{C}^3)z^{r+3}-(r\alpha^{-r-5}\varepsilon-\frac{r.r+5}{1.2}\alpha^{-r-6}.2\mathfrak{C}\delta+\gamma^2$

$+\frac{r.r+5.r+6}{1.2.3}\alpha^{-r-7}.3\mathfrak{C}^2\gamma-\frac{r.r+5..r+7}{1...4}\alpha^{-r-8}.\mathfrak{C}^4)z^{r+4}-\&c.$

Hence also,

107. $x^{-r}=\alpha^r z^{-r}+\frac{r}{r-1}\left|\frac{\alpha^{r-1}}{.}\right|'.z^{-r+1}+\frac{r}{r-2}\left|\frac{\alpha^{r-2}}{.}\right|''.z^{-r+2}$

$$+ \ldots + r\left|\frac{\alpha}{.}\right|^{r-1} z^{-1} + r\left|\frac{\log.\alpha}{.}\right|^{r} .z^{0} - \frac{r}{1}\left|\frac{\alpha^{-1}}{.}\right|^{r+1} .z - \frac{r}{2}\left|\frac{\alpha^{-1}}{.}\right|^{r+2} z^{2}$$

$$- \frac{r}{3}\left|\frac{\alpha^{-1}}{.}\right|^{r+3} z^{3} - \&c.$$

$$= \alpha^{r} z^{-r} + r\alpha^{r-2}\mathfrak{C} z^{-r+1} + (r\alpha^{r-3}\gamma + \frac{r.r-3}{1.2}\alpha^{r-4}\mathfrak{C}^{2})\, z^{-r+2}$$

$$+ (r\alpha^{r-4}\delta + \frac{r.r-4}{1.2}\alpha^{r-5}.2\mathfrak{C}\gamma + \frac{r.r-4.r-5}{1.2.3}\alpha^{r-6}\mathfrak{C}^{3})\, z^{-r+3}$$

$$+ (r\alpha^{r-5}\varepsilon + \frac{r.r-5}{1.2}\alpha^{r-6}.2\mathfrak{C}\delta + \gamma^{2}$$

$$+ \frac{r.r-5.r-6.r-7}{1.2.3.4}\alpha^{r-7}.\mathfrak{C}^{4})\, z^{-r+4} + \&c.$$

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108. Respecting this last series for the negative power of x, there are some remarkable things to be observed.

1. The first part of this series, as far as the term z^{-1}, where it seems to terminate, is the very same expression that was found before for the sum of the $-r$ powers, (or the r powers of the reciprocals,) of all the roots of the equation, when the polynome is definite.

2. Yet the whole series, continued indefinitely, expresses the $-r$ power of only the least root of the equation. For it is always the least root of a definite polynome that is found by reversion.

3. The intermediate term for z^0, between the affirmative and negative powers of z, appears singular. It may be deduced thus: By the law of derivation it would be $\frac{r}{o}\left|\frac{\alpha^o}{.}\right|^r . z^o$. But $\frac{1}{o}\left|\frac{\alpha^o}{.}\right|' = \alpha^{-1}\mathfrak{C} = \left|\frac{l.\alpha}{.}\right|'$. Consequently $\frac{r}{o}\left|\frac{\alpha^o}{.}\right|^r = r\left|\frac{l.\alpha}{.}\right|^r$.

4. Or the same term may be deduced, perhaps more satisfactorily, from a particular example. Thus, suppose $r = 2$; then, by the general developement above, we have,

$$x^{-2} = \alpha^2 . z^{-2} - 2\mathfrak{C}z^{-1} + (2\alpha^{-1}\gamma - \alpha^{-2}\mathfrak{C}^2)\, z^o,$$

for the first three terms, of which the last is evidently the developement of $2\left|\frac{l.\alpha}{.}\right|''$.

EXAMPLE.

Given the equation $7 = 15x - 2x^2$, or, $\frac{7}{15} = x - \frac{2}{15}x^2$ } of which the roots are 7 & $\frac{1}{2}$.

Here $z = \frac{7}{15}$, $\alpha = 1$, $\mathfrak{C} = -\frac{2}{15}$, $\gamma = o$, &c.

$x^2 = z^2 - 2\mathfrak{C}z^3 + 5\mathfrak{C}^2z^4 + \&c. = 0.218\&c. + 0.027\&c. + 0.004\&c.$

$= 0.25$, the square of the least root.

$S_{-2}=z^{-2}+2\mathfrak{C}z^{-1}=\frac{225}{49}-\frac{4}{7}=4\frac{1}{49}$, the sum of the squares of the reciprocals of the roots.

$x^{-2}=z^{-2}+2\mathfrak{C}z^{-1}-\mathfrak{C}^2z^0+2\mathfrak{C}^2z-3\mathfrak{C}^4z^2+$ &c.

$=\frac{225}{49}-\frac{4}{7}-\frac{4}{225}-\frac{16}{3375}-$ &c.

$=4$, the square of the reciprocal of the least root.

PROBLEM V.

109. Given $z=x(\alpha+\mathfrak{C}x+\gamma x^2+$ &c.$)^m$, to find $f(a+x)$.

Here $f(a+x)=fa+f'a.x+\frac{1}{2}f''a.x^2+\frac{1}{2.3}f'''a.x^3+$ &c. which series, (by Theorem, Article 99.) may be changed to,

$$fa+\alpha^{-m}f'a.s^mx+\tfrac{1}{2}\left|\frac{\alpha^{-2m}.f'a}{.}\right|'.s^{2m}x^2+\tfrac{1}{3}\left|\frac{\alpha^{-3m}.f'a}{.}\right|''.s^{3m}x^3$$

+ &c., supposing $p=1$, $q=1$: therefore, since $z=s^mx$,

$$f(a+x)=fa+\alpha^{-m}f'a.z+\tfrac{1}{2}\left|\frac{\alpha^{-2m}.f'a}{.}\right|'.z^2+\tfrac{1}{3}\left|\frac{\alpha^{-3m}.f'a}{.}\right|''.z^3$$

$$+\ldots\ldots\ldots\ldots+\frac{1}{n}\left|\frac{\alpha^{-mn}.f'a}{.}\right|^{n-1}.z^n+\text{&c.}$$

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110. Thus, if $z=\alpha x+\mathfrak{C}x^2+\gamma x^3+$ &c. and log. $(a+x)$ be required :

Since $m = 1$, $fa = l.a$, $f'a = a^{-1}$, $f''a = -a^{-2}$, &c. we have, by expanding the polynomial derivatives, as in Article 74,

$$\text{L.}(a+x) = l.a + \alpha^{-1}.a^{-1}.z + \tfrac{1}{2}\left(a^{-1}\left|\frac{\alpha^{-2}}{\cdot}\right|' - a^{-2}.\alpha^{-2}\right)z^2$$
$$+ \tfrac{1}{3}\left(a^{-1}\left|\frac{\alpha^{-3}}{\cdot}\right|'' - a^{-2}\left|\frac{\alpha^{-3}}{\cdot}\right|' + a^{-3}.\alpha^{-3}\right)z^3 + \tfrac{1}{4}\left(a^{-1}\left|\frac{\alpha^{-4}}{\cdot}\right|''' \right.$$
$$\left. - a^{-2}\left|\frac{\alpha^{-4}}{\cdot}\right|'' + a^{-3}\left|\frac{\alpha^{-4}}{\cdot}\right|' - a^{-4}.\alpha^{-4}\right)z^4 + \text{&c.}$$

And, if $a = 1$,

111. $$\text{L.}(1+x) = \alpha^{-1}z + \left(\left|\frac{\alpha^{-2}}{\cdot}\right|' - \alpha^{-2}\right)z^2 + \tfrac{1}{3}\left(\left|\frac{\alpha^{-3}}{\cdot}\right|'' - \left|\frac{\alpha^{-3}}{\cdot}\right|'\right.$$
$$\left. + \alpha^{-3}\right)z^3 + \tfrac{1}{4}\left(\left|\frac{\alpha^{-4}}{\cdot}\right|''' - \left|\frac{\alpha^{-4}}{\cdot}\right|'' + \left|\frac{\alpha^{-4}}{\cdot}\right|' - \alpha^{-4}\right)z^4 + \text{&c.}$$

But, if $a = o$, the terms of the series become infinite; $l.x$, therefore, cannot be found in terms of the powers of z.

Again, if $m = 1$, and sin. $(a + x)$ be required: Since fa = sin. a, and $f'a$ = cos. a, we have,

112. $$\text{Sin.}(a+x) = \text{sin.}\,a + \alpha^{-1}\,\text{cos.}\,a.z + \tfrac{1}{2}\left|\frac{\alpha^{-2}.\text{cos.}\,a}{\cdot}\right|'.z^2$$
$$+ \tfrac{1}{3}\left|\frac{\alpha^{-3}.\text{cos.}\,a}{\cdot}\right|''.z^3 + \tfrac{1}{4}\left|\frac{\alpha^{-4}.\text{cos.}\,a}{\cdot}\right|'''.z^4 + \text{&c.}$$

Here, if $a = o$, sin. $a = o$, and cos. $a = 1$.

Also, $\left|\frac{\alpha^{-2}.\cos.a}{.}\right|' = \alpha^{-2}\left|\frac{\cos.a}{.}\right|' + \left|\frac{\alpha^{-2}}{.}\right|' . \cos.\ a = \left|\frac{\alpha^{-2}}{.}\right|'.$

$\left|\frac{\alpha^{-3}\cos.a}{.}\right|'' = \alpha^{-3}\left|\frac{\cos.a}{.}\right|'' + \left|\frac{\alpha^{-3}}{.}\right|' . \left|\frac{\cos.a}{.}\right|' + \left|\frac{\alpha^{-3}}{.}\right|'' . \cos.\ a$

$= \left|\frac{\alpha^{-3}}{.}\right|'' - \frac{1}{2}\alpha^{-3}.$

For, $\left|\frac{\cos.\ a}{.}\right|' = -\sin.\ a\ .\ b = o,$

$\left|\frac{\cos.a}{.}\right|'' = -\sin.a\left|\frac{b}{.}\right|' - \frac{1}{2}\cos.a.b^2 = -\frac{1}{2}$(because $b=1$).

Hence,

113. Sin. $x = \alpha^{-1}.z + \frac{1}{2}\left|\frac{\alpha^{-2}}{.}\right|'.z^2 + \frac{1}{3}\left(\left|\frac{\alpha^{-3}}{.}\right|'' - \frac{1}{2}\alpha^{-3}\right)z^3$

$+ \frac{1}{4}\left(\left|\frac{\alpha^{-4}}{.}\right|''' - \frac{1}{2}\left|\frac{\alpha^{-4}}{.}\right|'\right)z^4 + \frac{1}{5}\left(\left|\frac{\alpha^{-5}}{.}\right|^{IV} - \frac{1}{2}\left|\frac{\alpha^{-5}}{.}\right|'' + \frac{1}{2.3.4}\alpha^{-5}\right)z^5$

$+ \frac{1}{6}\left(\left|\frac{\alpha^{-6}}{.}\right|^{V} - \frac{1}{2}\left|\frac{\alpha^{-6}}{.}\right|''' + \frac{1}{2.3.4}\left|\frac{\alpha^{-6}}{.}\right|'\right)z^6 +$ &c.

Lastly, if $m=1$, and e^{a+x}. be required, we shall have,

114. $e^{a+x} = e^a + \alpha^{-1}e^a.z + \frac{1}{2}\left|\frac{\alpha^{-2}.e^a}{.}\right|'.z^2 + \frac{1}{3}\left|\frac{\alpha^{-3}.e^a}{.}\right|''.z^3 +$ &c.

Hence, if $a = o$,

115. $e^x = 1 + \alpha^{-1}.z + \frac{1}{2}\left(\left|\frac{\alpha^{-2}}{.}\right|' + \alpha^{-2}\right)z^2 + \frac{1}{3}\left(\left|\frac{\alpha^{-3}}{.}\right|'' + \left|\frac{\alpha^{-3}}{.}\right|'\right.$

$\left. + \frac{1}{2}\alpha^{-3}\right)z^3 + \frac{1}{4}\left(\left|\frac{\alpha^{-4}}{.}\right|''' + \left|\frac{\alpha^{-4}}{.}\right|'' + \frac{1}{2}\left|\frac{\alpha^{-4}}{.}\right|' + \frac{1}{2.3}\alpha^{-4}\right)z^4 +$ &c.

As we cannot find the log. x by the method pointed out in the solution of this problem, it may be proper to shew how to obtain it.

From the equation, $z = x(\alpha + \mathfrak{C}x + \gamma x^2 + \delta x^3 + \text{&c.})$ we have,

$$\varphi\frac{z}{x} = \varphi\alpha + \left|\frac{\varphi\alpha}{\cdot}\right|' . x + \left|\frac{\varphi\alpha}{\cdot}\right|'' . x^2 + \left|\frac{\varphi\alpha}{\cdot}\right|''' . x^3 + \text{&c.}$$

$$= \varphi\alpha + \alpha^{-1}\left|\frac{\varphi\alpha}{\cdot}\right|' . z + \tfrac{1}{2}\left|\frac{\alpha^{-2}\left|\frac{\varphi\alpha}{\cdot}\right|'}{\cdot}\right|'' . z^2 + \tfrac{1}{3}\left|\frac{\alpha^{-3}\left|\frac{\varphi\alpha}{\cdot}\right|''}{\cdot}\right|''' . z^3 + \text{&c.}$$

(109.) or, since $\left|\frac{\varphi\alpha}{\cdot}\right|' = \varphi'\alpha \,.\, \mathfrak{C}$,

$$= \varphi\alpha + \varphi'\alpha \,.\, \alpha^{-1}\mathfrak{C}z + \tfrac{1}{2}\left|\frac{\varphi'\alpha.\alpha^{-2}\mathfrak{C}}{\cdot}\right|' z^2 + \tfrac{1}{3}\left|\frac{\varphi'\alpha.\alpha^{-3}\mathfrak{C}}{\cdot}\right|'' z^3$$

$$+ \ldots \frac{1}{n}\left|\frac{\varphi'\alpha.\alpha^{-n}\mathfrak{C}}{\cdot}\right|^{n-1} z^n + \text{&c.};$$

or, substituting $l.\alpha$ for $\varphi\alpha$, and α^{-1} for $\varphi'\alpha$,

$$l.z - l.x = l.\alpha + \alpha^{-2}\mathfrak{C}z + \tfrac{1}{2}\left|\frac{\alpha^{-3}\mathfrak{C}}{\cdot}\right|' z^2 + \tfrac{1}{3}\left|\frac{\alpha^{-4}.\mathfrak{C}}{\cdot}\right|'' . z^3 + \text{&c.}$$

Hence,

$$l\,.\,x = l\,.\,\alpha^{-1}\,.\,z - \alpha^{-2}\mathfrak{C}z - \tfrac{1}{2}\left|\frac{\alpha^{-3}\mathfrak{C}}{\cdot}\right|' . z^2 - \tfrac{1}{3}\left|\frac{\alpha^{-4}\mathfrak{C}}{\cdot}\right|'' . z^3$$

$$- \ldots - \frac{1}{n}\left|\frac{\alpha^{-n-1}.\mathfrak{C}}{\cdot}\right|^{n-1} . z^n - \text{&c.}$$

But $\left|\frac{\alpha^{-n}}{1}\right|' = -n\alpha^{-n-1}\mathfrak{C} = -n\left|\frac{\alpha^{-n-1}.\mathfrak{C}}{1}\right|^0$; therefore

$\left|\frac{\alpha^{-n}}{1}\right|^n = -n\left|\frac{\alpha^{-n-1}.\mathfrak{C}}{1}\right|^{n-1}$, or $\left|\frac{\alpha^{-n}}{.}\right|^n = -\left|\frac{\alpha^{-n-1}.\mathfrak{C}}{.}\right|^{n-1}$.

Consequently,

$$116.\ l.x = l.\alpha^{-1}.z - \alpha^{-2}\mathfrak{C}z + \tfrac{1}{2}\left|\frac{\alpha^{-2}}{.}\right|'' z^2 + \tfrac{1}{3}\left|\frac{\alpha^{-3}}{.}\right|''' .z^3 + \ldots \frac{1}{n}\left|\frac{\alpha^{-n}}{.}\right|^n z^n + \&c.$$

$$= l.\alpha^{-1}z - \alpha^{-2}\mathfrak{C}z + \tfrac{1}{2}(3\alpha^{-4}\mathfrak{C}^2 - 2\alpha^{-3}\gamma)z^2 - \tfrac{1}{3}(10\alpha^{-6}\mathfrak{C}^3 - 12\alpha^{-5}\mathfrak{C}\gamma + 3\alpha^{-4}\delta)\ z^3 + \tfrac{1}{4}\left(\tfrac{4.5.6.7}{2.3.4}\alpha^{-5}\mathfrak{C}^4 - \tfrac{4.5.6}{2.3}\alpha^{-7}.3\mathfrak{C}\gamma + \tfrac{4.5}{2}\alpha^{-6}.\overline{2\mathfrak{C}\delta+\gamma^2} - 4\alpha^{-5}\mathfrak{C}\right).z^4 - \&c.$$

Thus, for example, since $1 = .9999$ &c., if $z = 1$, $x = \frac{1}{10}$, $\alpha = \mathfrak{C} = \gamma = \&c. = 9$, we get,

$$\log. 10 = \log. 9 + \frac{1}{9} - \frac{1}{2}\cdot\frac{1}{9^2} + \frac{1}{3}\cdot\frac{1}{9^3} - \frac{1}{4}\cdot\frac{1}{9^4} + \&c.$$

SECTION XX.

OF EQUATIONS INVOLVING SOME GENERAL FUNCTION OF THE UNKNOWN QUANTITY.

PROBLEM I.

117. GIVEN $z = a + bfz$, to find z, or any function of z.

Because $z = a + bfz = a + b \cdot f(a + bfz)$
$= a + fa \cdot b +$ &c.

Let $z = a + fa \cdot b + A \cdot b^2 + B \cdot b^3 + C \cdot b^4 +$ &c.

then (57.) $fz = fa + \left|\frac{fa}{\cdot}\right|' \cdot b + \left|\frac{fa}{\cdot}\right|'' \cdot b^2 + \left|\frac{fa}{\cdot}\right|''' \cdot b^3 +$ &c.

But, by equating the values of z above, we have also,

$$fz = fa + A \cdot b + B \cdot b^2 + C \cdot b^3 + \text{&c.}$$

Hence $A = \left|\frac{fa}{\cdot}\right|'$, $B = \left|\frac{fa}{\cdot}\right|''$, $C = \left|\frac{fa}{\cdot}\right|'''$, &c. the values of which polynomial derivatives are easily found by Table II. (67.), by putting $\epsilon = fa$, $\gamma = A$, $\delta = B$, &c. Thus,

$A = f'a \,.\, fa.$ $\qquad = \frac{1}{2}(f^2a)'.$

$B = f'a \,.\, A + \frac{1}{2}f''a \,.\, f^2a.$ $\qquad = \frac{1}{2.3}(f^3a)''.$

$C = f'a.B + \frac{1}{2}f''a.2Afa + \frac{1}{2.3}f'''a.f^3a = \frac{1}{2.3.4}(f^4a)'''.$

&c. $\qquad$ &c.

Consequently,

118. $$z = a + fa \,.\, b + \tfrac{1}{2}(f^2a)' \,.\, b^2 + \frac{1}{2.3}(f^3a)'' \,.\, b^3 + \frac{1}{2.3.4}(f^4a)'''.b^4 + \&c.$$

and

$$\varphi z = \varphi a + \varphi' a \,.\, fab + (\varphi' a \,.\, (f^2a)' + \varphi'' a \,.\, f^2a)\tfrac{1}{2}b^2 + (\varphi' a.(f^2a)'' + 3\varphi'' a.fa.(f^2a)' + \varphi''' a.f^2a)\frac{1}{2.3}b^3 + \&c.$$

that is,

119. $$\varphi z = \varphi a + \varphi' a \,.\, fab + \tfrac{1}{2}(\varphi' a \,.\, f^2a)' \,.\, b^2 + \frac{1}{2.3}(\varphi' a \,.\, f^2a)'' \,.\, b^3 + \&c.$$

The theorems (118. 119.) found by M. La Grange for the solution of this problem being extensively useful, we shall illustrate the application of them by a few examples.

EXAMPLE 1.

Given $\lambda = \dfrac{1 - \sqrt{1 - \varepsilon^2}}{\varepsilon}$, to find λ^n by a series.

We have, by reduction, $\lambda = \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon\lambda^2$, an equation similar to that in the problem. Hence, putting $c = \frac{1}{2}\varepsilon$, we have $z = \lambda$, $a = c$, $b = c$, $fz = \lambda^2$, $fa = c^2$, $\varphi z = \lambda^n$, and $\varphi a = c^n$; therefore, by Theorem 119.,

$$\lambda^n = c^n + nc^{n+2} + \frac{n.n+3}{2}c^{n+4} + \frac{n.n+3.n+5}{2.3}c^{n+6} + \&c.$$

La Place Mec. Cel. vol. i. p. 180.

EXAMPLE 2.

Given (m) the mean anomaly of a planet, and (ε) the eccentricity of its orbit, to find (x) the eccentric anomaly from the following equation:

$$x = m - \varepsilon \sin. x.$$

Comparing this with the general equation above, we have only to substitute x for z, sin. x for fz, m for a, and $-\varepsilon$ for b, and we get,

$$x = m - \varepsilon \sin. m + \tfrac{1}{2}\varepsilon^2 (\sin.^2 m)' - \frac{1}{2.3}\varepsilon^3 (\sin.^3 m)''$$

$$+ \frac{1}{2.3.4}\varepsilon^4 (\sin.^4 m)''' - \&c.$$

$$= m - \varepsilon \sin. m + \tfrac{1}{2}\varepsilon^2 \sin. 2m - \frac{1}{2.2^2}\varepsilon^3 (3 \sin. 3m - \sin. m)$$

$+\frac{1}{2.3.2^3}\varepsilon^4(16\sin. 4m - 8\sin. 2m) - \frac{1}{2.3.4.2^4}\varepsilon^5(125\sin. 5m - 81\sin. 3m + 2\sin. m) + \&c.$

La Place Mec. Cel. vol. i. p. 179.

For $(\sin.^3 m)'' = \frac{1}{4}(3\sin. m - \sin. 3m)'' = \frac{1}{4}(3\cos. m - 3\cos. 3m)' = \frac{1}{4}(-3\sin. m + 9\sin. 3m)$. &c. &c.

This expression for x may be reduced to a more convenient form: Thus,

$$x = m - \sin. m\left(\varepsilon - \frac{1}{1.2.2^2}\varepsilon^3 + \frac{1}{1.2.3.1.2.2^4}\varepsilon^5 - \frac{1}{1..4.1.2.3.2^6}\varepsilon^7 + \frac{1}{1..5.1..4.2^8}\varepsilon^9 - \&c.\right)$$

$$+ \sin. 2m\left(\frac{2}{1.2.2}\varepsilon^2 - \frac{2^3}{1.2.3.2^3}\varepsilon^4 + \frac{2^5}{1..4.1.2.2^5}\varepsilon^6 - \frac{2^7}{1..5.1..3.2^7}\varepsilon^8 + \&c.\right)$$

$$- \sin. 3m\left(\frac{3^2}{1.2.3.2^2}\varepsilon^3 - \frac{3^4}{1..4.2^4}\varepsilon^5 + \frac{3^6}{1..5.1.2.2^6}\varepsilon^7 - \frac{3^8}{1..6.1.2.3.2^8}\varepsilon^9 + \&c.\right)$$

$$+ \sin. 4m\left(\frac{4^3}{1..4.2^3}\varepsilon^4 - \frac{4^5}{1..5.2^5}\varepsilon^6 + \frac{4^7}{1..6.1.2.2^7}\varepsilon^8 - \frac{4^9}{1..7.1.2.3.2^9}\varepsilon^{10} + \&c.\right)$$

$$- \sin. 5m\left(\frac{5^4}{1..5.2^4}\varepsilon^5 - \frac{5^6}{1..6.2^6}\varepsilon^7 + \frac{5^8}{1..7.1.2.2^8}\varepsilon^9 - \frac{5^{10}}{1..8.1.2.3.2^{10}}\varepsilon^{11} + \&c.\right)$$

&c. &c.

Cor. Hence we have also $\sin. x$, $\left(\text{being} = \frac{m - x}{\varepsilon}\right)$.

EXAMPLE 3.

The same things being given as in the preceding example, to find the radius vector r from the following equations :

$$x = m - \varepsilon \sin. x.$$
$$r = 1 + \varepsilon \cos. x.$$

From the first equation, by Theorem 119., we find,

$$\cos. x = \cos. m + \varepsilon \sin.^2 m - \tfrac{1}{2}\varepsilon^2 (\sin.^2 m)' + \frac{1}{2.3}\varepsilon^3 (\sin.^4 m)'' - \&c.$$

Therefore,

$$r = 1 + \varepsilon \cos. m + \varepsilon^2 \sin.^2 m - \tfrac{1}{2}\varepsilon^2 (\sin.^2 m)' + \frac{1}{2.3}\varepsilon^4 (\sin.^4 m)'' - \&c.$$

$$= \overline{1+\tfrac{1}{2}\varepsilon^2} + \varepsilon \cos. m - \tfrac{1}{2}\varepsilon^2 \cos. 2m + \tfrac{3}{8}\varepsilon^3 (\cos. 3m - \cos. m) - \tfrac{1}{3}\varepsilon^4 (\cos. 2m - \cos. 4m) + \tfrac{1}{384}\varepsilon^5 (125 \cos. 5m - 135 \cos. 3m + 10 \cos. m) - \&c.$$

EXAMPLE 4.

Given $z = a + bz^3$, to find z.

Here $fa = a^3$, and (118.)

$$z = a + a^3 b + 3a^5 b^2 + 12a^7 b^3 + 55a^9 b^4 + 273a^{11} b^5 + \&c.$$

The very same may be found by the reversion of series (103.).

Thus, if $z^3 - 14z + 8 = 0$, or $z = \frac{4}{7} + \frac{1}{14}z^3$,
we find $z = 0.5857$, &c.

And, if $z^3 - 5z - 2 = 0$, or $z = -\frac{2}{5} + \frac{1}{5}z^3$,
we find $z = -0.41423$, &c.

The least value of z, (whether affirmative or negative,) is found by this method, (108. 2.).

PROBLEM II.

120. Given $b = x \,.\, f(a + x)$, to find $\varphi(\alpha + x)$.

Put $s = f(a+x) = fa + f'a.x + \frac{1}{2}f''a.x^2 + \frac{1}{2.3}f'''a.x^3 + \&c.$

then $sx = b$, and

$$\varphi(\alpha + x) = \varphi\alpha + \varphi'\alpha.x + \tfrac{1}{2}\varphi''\alpha.x^2 + \frac{1}{2.3}\varphi'''\alpha.x^3 + \&c.$$

$$= \varphi\alpha + (fa)^{-1}.\varphi'\alpha.b + \tfrac{1}{2}\left|\frac{(fa)^{-2}.\varphi'\alpha}{\cdot}\right|'.b^2 + \tfrac{1}{3}\left|\frac{(fa)^{-3}.\varphi'\alpha}{\cdot\ \cdot}\right|''.b^3$$

$$+ \ldots\ldots + \left|\frac{(fa)^{-n}.\varphi'\alpha}{\cdot}\right|^{n-1}.b^n + \&c.$$

Cor. If $b = (z - a) \,.\, fz$, and φz be required: by putting $a + x$ for z, we have $b = x \,.\, f(a + x)$; whence φz, or $\varphi(a + x)$, may be found, by making $\alpha = a$ in the expression above.

PROBLEM III.

121. Given $b = \frac{x}{f(a+x)}$, to find $\varphi(\alpha + x)$.

Here we have only to substitute fa in place of $(fa)^{-1}$ in the preceding article, and we get,

$$\varphi(\alpha+x) = \varphi\alpha + fa.\varphi'\alpha.b + \tfrac{1}{2}\left|\frac{f^2a.\varphi'\alpha}{\cdot}\right|'.b^2 + \tfrac{1}{3}\left|\frac{f^2a.\varphi'\alpha}{\cdot}\right|''.b^3$$
$$+ \ldots\ldots\ldots + \frac{1}{n}\left|\frac{f^2a.\varphi'\alpha}{\cdot}\right|^{n-1}.b^n + \&c.$$

122. Cor. 1. If $bfz = z - a$, and φz be required: by making $z = a + x$, we have $b = \frac{x}{f(a+x)}$; whence φz, or $\varphi(a + x)$, may be found as above, putting $\alpha = a$.

The expression for φz, thus obtained, is the same with that, found by another process, in the solution of Problem I. (119.). But it may be added here, that, if $\varphi(r + z,)$ instead of φz, be required, we have only to make $\alpha = a + r$, instead of $\alpha = a$.

123. Cor. 2. In this and the preceding Problem, we may find φx in place of $\varphi(\alpha + x)$, by putting

$\alpha = o$, and $\alpha' = 1$, after making the requisite derivations, provided $\varphi\alpha$ be not such a function, that, when $\alpha = o$, $\varphi\alpha$ and its derivatives either vanish or become infinite.

Thus, if $b.f(a+x) = x$, by making $\varphi(\alpha+x) = x$, we get,

$$x = fa.b + \tfrac{1}{2}\left|\frac{f^2a}{.}\right|'.b^2 + \tfrac{1}{3}\left|\frac{f^2a}{.}\right|''.b^3 + \ldots + \frac{1}{n}\left|\frac{f^2a}{.}\right|^{n-1}.b^n + \&c.$$

$$= fa\,.\,b + \tfrac{1}{2}(f^2a)'.b^2 + \frac{1}{2.3}(f^2a)''\,.\,b^3 + \ldots$$

$$+ \frac{1}{2\ldots n}(f^2a)^{n-1}\,.\,b^n + \&c.$$

PROBLEM IV.

124. Given $x = \mathrm{F}(a + bfx)$, to find φx.

Put $z = a + bfx$, then $x = \mathrm{F}z$, $fx = f\mathrm{F}z$, and $\varphi x = \varphi\mathrm{F}z$. Hence $z = a + b\,.\,f\mathrm{F}z$; from which equation, by Problem I., we may find any other function of z, as $\varphi\mathrm{F}z$ $(= \varphi x)$. Thus, putting $\backslash f$ for $f\mathrm{F}$, and $\backslash\varphi$ for $\varphi\mathrm{F}$,

$$\varphi x = \backslash\varphi a + \backslash\varphi' a\,.\,\backslash f a\,.\,b + \tfrac{1}{2}(\backslash\varphi' a\,.\,\backslash f^2 a)'\,.\,b^2$$
$$+ \frac{1}{2.3}(\backslash\varphi' a\,.\,\backslash f^2 a)''\,.\,b^3 + \&c.$$

Otherwise,

Put $v = bfx$, then we shall have $v = bf\mathrm{F}(a+v)$; from which equation, by Cor. 2. Problem III., we find,

$$v = \text{.}fa.b + \tfrac{1}{2}(\text{.}f^2a)'.b^2 + \frac{1}{2.3}(\text{.}f^3a)''.b^3 + \frac{1}{2.3.4}(\text{.}f^4a)'''.b^4 + \&c.$$

$$x = \mathrm{F}(a + \text{.}fa.b + \tfrac{1}{2}(\text{.}f^2a)'.b^2 + \frac{1}{2.3}(\text{.}f^3a)''.b^3 + \&c.)$$

and

$$\varphi x = \text{.}\varphi(a + \text{.}fa.b + \tfrac{1}{2}(\text{.}f^2a)'.b^2 + \frac{1}{2.3}(\text{.}f^3a)''.b^3 + \&c.)$$

From comparing these values of φx, we deduce the following Theorem :

125. $$\varphi(\alpha + \xi z + \tfrac{1}{2}(\xi^2)'.z^2 + \frac{1}{2.3}(\xi^3)''.z^3 + \frac{1}{2.3.4}(\xi^4)'''.z^4 + \&c.)$$
$$= \varphi\alpha + \varphi'\alpha.\xi z + \tfrac{1}{2}(\varphi'\alpha.\xi^2)z^2 + \frac{1}{2.3}(\varphi'\alpha.\xi^3)''.z^3 + \&c.$$

SECTION XXI.

TO EXPRESS ANY FUNCTION OF A QUANTITY BY A SERIES IN TERMS OF ANY OTHER FUNCTION OF THE SAME, AND OF SOME ARBITRARY QUANTITY.

126. LET Fz and fz be the two functions of z; it is required to express the former by a series in terms of the latter, and of the like functions of the arbitrary quantity a.

Since $Fz = F(a+z-a)$, and $fz = f(a+z-a)$, by making $z-a = x$, we shall have,

$$fz = f(a+x) = fa + f'a.x + \tfrac{1}{2}f''a.x^2 + \tfrac{1}{2.3}f'''a.x^3 + \text{&c.}$$

and

$$fz - fa = x(f'a + \tfrac{1}{2}f''a.x + \tfrac{1}{2.3}f'''a.x^2 + \text{&c.}) = xs.$$

But

$$Fz = F(a+x) = Fa + F'a.x + \tfrac{1}{2}F''a.x^2 + \tfrac{1}{2.3}F'''a.x^3 + \text{&c.}$$

Therefore,

$$Fz = Fa + (f'a)^{-1} \cdot F'a\,(fz - fa)$$

$$+\tfrac{1}{2}\left|\frac{(f'a)^{-2}.\mathrm{F}'a}{.}\right|'(fz-fa)^2+\ldots\ldots$$
$$+\frac{1}{n}\left|\frac{(f'a)^{-n}.\mathrm{F}'a}{.}\right|^{n-1}.(fz-fa)^n+\&c.$$

127. Cor. 1. By substituting x in place of a, and $x+i$ in place of z, we get,

$$fz-fa=f(x+i)-fx=\Delta \text{ (suppose)};$$

then

$$\mathrm{F}(x+i)=\mathrm{F}x+(f'x)^{-1}.\mathrm{F}'x.\Delta+\tfrac{1}{2}\left|\frac{(f'x)^{-2}.\mathrm{F}'x}{.}\right|'.\Delta^2$$
$$+\ldots\ldots\frac{1}{n}\left|\frac{(f'x)^{-n}.\mathrm{F}'x}{.}\right|^{n-1}.\Delta^n+\&c.$$

Which series, when $fx=x$, and, consequently $\Delta=i$, becomes the developement of $\mathrm{F}(x+i)$ in Section I.

128. Cor. 2. If $fa=o$,

$$\mathrm{F}z=\mathrm{F}a+(\mathrm{F}'a)^{-1}.\mathrm{F}'a.fz+\tfrac{1}{2}\left|\frac{(f'a)^{-2}.\mathrm{F}'a}{.}\right|'.f^2z$$
$$+\tfrac{1}{3}\left|\frac{(fa)^{-3}.\mathrm{F}'a}{.}\right|''.f^3z+\&c.$$

If $fz=o$,

$$\mathrm{F}z=\mathrm{F}a-(f'a)^{-1}.\mathrm{F}'a.fa+\tfrac{1}{2}\left|\frac{(f'a)^{-2}.\mathrm{F}'a}{.}\right|'.f^2a$$
$$-\tfrac{1}{3}\left|\frac{(f'a)^{-3}.\mathrm{F}'a}{.}\right|''.f^3a+\&c.$$

If, for example, $fz = o$, and a be assumed nearly equal to one of the roots of that equation, we shall have,

$$z' = a^r - (f'a)^{-1} \cdot (a^r)' \cdot fa + \tfrac{1}{2} \left| \frac{(f'a)^{-2} \cdot (a^r)'}{\cdot} \right|' \cdot f^2a$$

$$- \tfrac{1}{3} \left| \frac{(f'a)^{-3} \cdot (a^r)'}{\cdot} \right|'' \cdot f^3a + \&c. ;$$

which, when $r = 1$, and, therefore, $(a^r)' = 1$, becomes the series for z that is found by the reversion of

$$o = fa + f'a \cdot x + \tfrac{1}{2} f''a \cdot x^2 + \&c. ;$$

or, $- fa = f'a \cdot x + \tfrac{1}{2} f''a \cdot x^2 + \frac{1}{2.3} f'''a \cdot x^3 + \&c.$

supposing $z = a + x$.

PART II.

INTEGRATION;

OR, THE INVERSE METHOD OF ANALYSIS.

INTRODUCTION

TO THE

THEORY

OF

ANALYTICAL FUNCTIONS.

SECTION I.

OF FINDING THE PRIMITIVE FUNCTION.

1. HAVING shewn how to find the prime function of a given primitive, and all other derivative functions, we proceed to treat of the inverse method, termed INTEGRATION, viz. that of finding the primitive of a given prime function; or rather, of finding the variable part of the primitive. For, if a constant quantity make part of the primitive, as no vestige of it appears in the derivative function, it must remain undetermined. To find the variable part, however, is sufficient, as the constant part may be ascertained from the nature of the problem, for the solution of which, the complete primitive is required.

The inverse method is more difficult, as well as more extensively useful, than the direct; and can only be pursued, as far as it is practicable, by rules reversing the preceding operations; but, in most cases, recourse must be had to methods of approximation.

The prime function, for example, of y^n, a given power of y, being $ny^{n-1}.y'$, it is easy to deduce this rule for finding the primitive, in similar cases, from the prime function, viz. Add 1 to the index of the variable quantity, and divide the function so changed by the increased index multiplied into the prime function of the root; for $\frac{ny^{n-1+1}y'}{n-1+1.y'} = y^n$. And this rule holds, whether the index be affirmative or negative, integral or fractional. There is only one case in which it fails; and that is, when the index of the variable is -1: for, in this case, the primitive of $y^{-1}.y'$, $\left(\text{or } \frac{y'}{y}\right)$, becomes, according to the rule, $\left(\frac{1}{o}, \text{ or }\right)$ infinite. We should therefore have been led to conclude that $\frac{y'}{y}$ had no finite primitive, had not the preceding investigation discovered to us its primitive (log. y) among a different kind of functions.

We shall now set down some formulæ, to serve in place of rules, of three different kinds of functions, distinguished by the terms Algebraic, Logarithmic, and Circular; and by a few examples under each, point out the method of transforming (if possible) given prime functions into such as shall be similar to the formulæ, so that their primitives may be discovered; referring such cases as are more difficult, or altogether impracticable by this method, to be treated of more fully afterwards, when some preparatory subjects shall have been discussed.

The integral or primitive of any function is marked by this sign $\int$ placed before it; thus, $\int y' = y$, $\int(2yy' + 2zz') = y^2 + z^2$, &c. And it is to be observed, that $\int ay' = a\int y'$, $\int c^2xy' = c^2\int xy'$, &c.

2. I. ALGEBRAIC FUNCTIONS.

1. $\int(ax' \pm by') = ax \pm by.$

2. $\int ax^n x' = \frac{a}{n+1}x^{n+1}.$

3. $\int(xy' + yx') = xy.$

4. $\int\left(\frac{x'}{y} - \frac{xy'}{y^2}\right) = \frac{x}{y}.$

EXAMPLE 1.

Required the primitive of $-\frac{xx'}{\sqrt{a^2 - x^2}}$.

Put $a^2 - x^2 = z^2$; then $-xx' = zz'$, and $-\frac{xx'}{\sqrt{a^2-x^2}} = \frac{zz'}{z} = z'$; therefore $\int -\frac{xx'}{\sqrt{a^2-x^2}} = z = \sqrt{a^2-x^2}$.

EXAMPLE 2.

Let $\frac{a^2-2x^2}{\sqrt{a^2-x^2}} \cdot x' \ (=y')$ be the given prime function.

Putting $a^2-x^2=z^2$, as before, we have $y' = \frac{z^2-x^2}{z} \cdot x'$

$= zx' - \frac{x^2x'}{z} =$ (because $-xx' = zz'$) $zx' + xz'$.

Hence $y = \int(zx' + xz') = xz = x\sqrt{a^2-x^2}$.

EXAMPLE 3.

Given $y' = (a^2-x^2)^2 \cdot xx'$, to find the primitive y; supposing $y = o$, when $x = o$.

By the second formula, $y = -\frac{(a^2-x^2)^3}{6}$; which, when $x = o$, becomes $-\frac{a^6}{6}$; therefore $\frac{a^6-(a^2-x^2)^3}{6}$ is the correct primitive required.

Or thus,

Since $(a^2-x^2)^2 \cdot xx' = a^4xx' - 2a^2x^3x' + x^5x'$;

$y = \frac{1}{2}a^4x^3 - \frac{1}{2}a^2x^4 + \frac{1}{6}x^6$; which, when $x = o$, also vanishes: the primitive, therefore, found in this way needs no correction, and is, indeed, the same with the corrected primitive found above.

EXAMPLE 4.

Given $y' = -\frac{2nx \cdot (1-x^2)^{n-1}}{mx^{p-1}} - \frac{px^{p-1} \cdot (1-x^2)^n}{mx^{p+1}}$.

Put $z = (1-x^2)^n$, and $v = x^p$; then

$$z' = -2nxx'(1-x^2)^{n-1}.$$
$$v' = px^{p-1} \cdot x'.$$

Hence $y' = \frac{z'}{mv} - \frac{pzx'}{mvx}$; but $px' = \frac{xv'}{v}$, and $\frac{x'}{x} = \frac{v'}{pv}$;

therefore, $y' = \frac{z'}{mv} - \frac{zv'}{mv^2} = \frac{1}{m}\left(\frac{z'}{v} - \frac{zv'}{v^2}\right)$.

Consequently, $y = \frac{1}{m} \cdot \frac{z}{v} = \frac{(1-x^2)^n}{mx^p}$.

EXAMPLE 5.

To find the primitive of $y' = ax^{n-1}x'$, supposing $y = o$, when $x = b$.

It is evident that $\frac{ax^n}{n}$ is the variable part of the primitive, which, when $x = b$, becomes $\frac{ab^n}{n}$; to make

If $z = l \cdot x$, then $z' = \frac{x'}{x}$, and $y' = zz'$; therefore

$$y = \tfrac{1}{2} z^2 = \tfrac{1}{2} (l \cdot x)^2.$$

EXAMPLE 3.

Given $y' = l \cdot x \cdot x^{n-1} x'$, to find y.

Put $z = \frac{1}{n} x^n$, then $z' = x^{n-1} x'$, and $y' = l \cdot x \cdot z'$.

Hence $y = \int l \cdot x \cdot z' = l \cdot x \cdot z - \int z \cdot \frac{x'}{x} = \frac{1}{n} x^n \cdot l \cdot x$

$$- \int \frac{1}{n} x^{n-1} x' = \frac{1}{n} x^n \cdot l \cdot x - \frac{1}{n^2} x^n = \frac{1}{n^2} x^n (l \cdot x^n - 1).$$

EXAMPLE 4.

Given $y' = \frac{2x'}{(1-x)\sqrt{x}}$, to find y.

Put $x^2 = x$, then $y' = \frac{4z'}{(1-z^2)z} = \frac{4z'}{1-z^2}$; therefore

$$y = 2l \cdot \frac{1+z}{1-z} = 2l \cdot \frac{1+\sqrt{x}}{1-\sqrt{x}}.$$

EXAMPLE 5.

Given $y' = \frac{x'}{x\sqrt{x}} \cdot l \cdot \frac{1}{1-x}$, to find y.

Put $z = -2x^{-\frac{1}{2}}$, then $z' = \frac{x'}{x\sqrt{x}}$, and if $c = l \cdot \frac{1}{1-x}$

$y'=vz'=vz-\int zv'=\frac{2}{\sqrt{x}}.l.\overline{1-x}-2l.\frac{1+\sqrt{x}}{1-\sqrt{x}}$, (by Example 4.)

EXAMPLE 6.

Given $y'=\frac{x'\sqrt{a^2+x^2}}{a}$, to find y.

Because $\frac{x'\sqrt{a^2+x^2}}{a}=\frac{x'(a^2+x^2)}{a\sqrt{a^2+x^2}}=\frac{\frac{1}{2}ax'}{\sqrt{a^2+x^2}}+\frac{(\frac{1}{2}a^2+x^2).x'}{a\sqrt{a^2+x^2}}$,

therefore $y=\frac{1}{2}al.(x+\sqrt{a^2+x^2})+\frac{x\sqrt{a^2+x^2}}{2a}$.

4. III. CIRCULAR FUNCTIONS.

Let r, s, c, t, v, f be the radius, sine, cosine, tangent, versed sine, and secant, of the circular arc z: Then,

$$z=\int\frac{rs'}{\sqrt{r^2-s^2}}=\int\frac{-rc'}{\sqrt{r^2-c^2}}=\int\frac{r^2t'}{r^2+t^2}=\int\frac{rv'}{\sqrt{2rv-v^2}}$$
$$=\int\frac{rv'}{s}=\int\frac{r^2f'}{f\sqrt{f^2-r^2}};$$

or,

If 1, $\frac{s}{r}$, $\frac{c}{r}$, $\frac{t}{r}$, $\frac{v}{r}$, $\frac{f}{r}$ be the similar lines appertaining to

the arc $\frac{z}{r}$, the radius being 1; then,

$$\frac{z}{r}=\int\frac{s'}{\sqrt{r^2-s^2}}=\int\frac{-c'}{\sqrt{r^2-c^2}}=\int\frac{rf'}{r^2+f^2}=\int\frac{v'}{\sqrt{2rv-v^2}}$$

$$=\int\frac{v'}{s}=\int\frac{rf'}{f\sqrt{f^2-r^2}}$$

EXAMPLE 1.

Required the primitive of $\frac{bx'}{c^2+x^2}$.

$$\int\frac{bx'}{c^2+x^2}=\frac{b}{c}\int\frac{cx'}{c^2+x^2}=\frac{b}{c}\times \text{ arc (whose tan. is } \frac{x}{c} \text{ to rad. 1).}$$

EXAMPLE 2.

Given $y'=x'\sqrt{a^2-x^2}$, to find y.

$$\text{Because } x'\sqrt{a^2-x^2}=\frac{2x'(a^2-x^2)}{2\sqrt{a^2-x^2}}=\frac{a^2x'}{2\sqrt{a^2-x^2}}$$

$$+\frac{x'(a^2-2x^2)}{2\sqrt{a^2-x^2}};$$

therefore $y=\frac{1}{2}a^2\cdot\text{arc}\left(\sin.=\frac{x}{a}\right)+\frac{1}{2}x\sqrt{a^2-x^2}$. (2. Example 2.)

EXAMPLE 3.

Given $y'=\frac{(ax+b)x'}{c^2+x^2}$, to find y.

The given function being resolved into $\frac{axx'}{c^2+x^2}+\frac{bx'}{c^2+x^2}$, its primitive is evidently $\frac{1}{2}a.l.(c^2+x^2)+\frac{b}{c}.\text{arc}\left(\tan.\frac{x}{c}\right)$.

EXAMPLE 4.

Given $y'=\frac{x'}{a+b\cos.x}$, to find y.

Put $\cos.x=\frac{1-z^2}{1+z^2}$; let $\overline{a+b}=s$, and $\overline{a-b}=d$; then $\sin.x=\frac{2z}{1+z^2}$, and, if $a>b$, $y'=\frac{2z'}{s+dz^2}$; but if $a<b$, $y'=\frac{2z}{s-dz^2}$, the former is a circular, the latter a logarithmic function. We have, accordingly,

$$y=\frac{2}{\sqrt{sd}}.\text{arc}\left(\tan.\frac{\sqrt{d}}{\sqrt{s}}.z\right),\text{ or, }y=\frac{1}{\sqrt{sd}}.l.\left(\frac{\sqrt{sd}+dz}{\sqrt{sd}-dz}\right).$$

SECTION II.

QUADRATURE OF CURVES, AND CUBATURE OF SOLIDS.

5. In the general equation of curves, $y=fx$, y is a rectangular ordinate corresponding to the abscis x, of

which it is a given function. The space bounded by the curve, ordinate, and absciss, may likewise be supposed a function of the same absciss x, and denoted by Fx. Then $F(x+i)-Fx$ will be the space contained by the two ordinates, $f(x+i)$, and fx, the small portion i of the axis, and the corresponding portion of the curve. This space, it is evident, whether the ordinates increase or decrease, is of an intermediate magnitude between the rectangles $i \,.\, fx$ and $i \,.\, f(x+i)$. But $F(x+i) - Fx = i \,.\, F'x + \frac{i^2}{2} \,.\, F''(x+u)$; which latter quantity, therefore, must be between the said rectangles $i.fx$ and $i.f(x+i)$, and must consequently differ less from one of them, than they do from each other; that is, $i \,.\, (F'x - fx) + \frac{i^2}{2} F''(x+u) \lessgtr i^2.f'(x+i)$, however small the increment i may be. But this is impossible, unless the term $i \,.\, (F'x - fx)$ vanish. *Hence,* $F'x - fx = o$, or $F'x = f$

Thus, the area and being both fun
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Supposing, area, w
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considered

only to substitute $\frac{v'}{x'}$ in place of v' in the preceding equation, and we get $\frac{v'}{x'} = y$, or $v' = yx'$.

To find, therefore, the area which any curve, with its ordinate and absciss, contains, we must multiply the ordinate by the prime function of the absciss, and, considering the product as a derivative function, find its primitive; then add to that primitive, (if necessary,) some constant quantity, determined by the nature of the problem.

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6. If the function φx denote the area of the section of a solid, made perpendicular to the axis, it may be proved in the same manner, that the content of the solid will be expressed by the primitive function of φx; or, more generally, by the primitive of $\varphi x \,.\, x'$.

EXAMPLE 1.

To find the area of the common parabola, and the content of the solid generated by the revolution of the same about its axis.

From the equation $y^2 = px$, we have the ordinate $y = \sqrt{p} \,.\, x^{\frac{1}{2}}$, which being considered as a prime func-

tion, its primitive $\frac{2}{3}\sqrt{p} \cdot x^{\frac{3}{2}} = \frac{2}{3}\sqrt{px} \cdot x = \frac{2}{3}xy$ is the area; being $\frac{2}{3}$ of the rectangle under the ordinate and absciss.

Again, if $c =$ area of the circle whose radius is 1, cy^2 will be the area of the circular section of the conoid formed by the revolution of the parabola about its axis. Now $cy^2 = cpx$, the primitive function whereof is $\frac{1}{2}cpx^2 = cy^2 \cdot \frac{1}{2}x$, which therefore is the solidity of the conoid; being half the cylinder of the same base and altitude.

EXAMPLE 2.

Given the parabolic equation $y^{m+n} = p^m x^n$, to find the area of the curve, and solidity of the conoid.

$$\text{Area of the parabola, } = \frac{m+n}{m+2n} \cdot xy.$$

$$\text{Solidity of the conoid, } = cy^2 \cdot \frac{m+n}{m+3n} x.$$

EXAMPLE 3.

To find the solidity of any segment of a sphere.

Let x and y be the altitude of the segment and the radius of its base, or the abscis and ordinate of the circle whose radius is r; then cy^2 is the area of the base; and the equation of the curve being $y^2 = 2rx - x^2$,

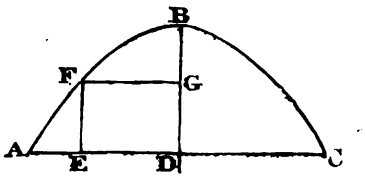
we have $cy^2 = 2crx - cx^2$, of which, therefore, the primitive $crx^2 - \frac{1}{3}cx^3$ is the solidity of the segment.

This expression, when $x = r$, becomes $\frac{2}{3}cr^3$ = solidity of the hemisphere $= \frac{2}{3}$ cylinder of the same base and altitude.

EXAMPLE 4.

To find the solidity of a parabolic spindle.

Let the segment ABC of a parabola, revolve about its base AC, the solid generated thereby is called a spindle. Let EF be perpendicular to AC from any point F in the curve; put AE$=x$, EF$=y$, as usual,



also AD $= a$, and BD $= B$; then by the property of the curve, $b : b - y = a^2 : (a - x)^2$, hence, $b : y = a^2 : 2ax - x^2$, and $cy^2x' = \frac{cb^2}{a^4}(2ax - x^2)^2 \cdot x'$ $= \frac{cb^2}{a^4} \cdot (4a^2x^2x' - 4a^2x^3x' + x^4x')$, whereof the primitive is $\frac{cb^2}{a^4}(\frac{4}{3}a^2x^3 - ax^4 + \frac{1}{5}x^5)$ = solidity of the part described by the revolution of AEF, which, when $x = a$, becomes $\frac{8}{15}cb^2a = \frac{1}{2}$ parabolic spindle $= \frac{8}{15}$ cylinder of the same base and altitude.

EXAMPLE 5.

Required the quadrature of the curve, whose equation is $x^4 - a^2x^2 + a^2y^2 = o$.

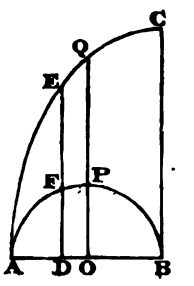
Here the ordinate $y = \frac{x\sqrt{a^2-x^2}}{a}$, of which the variable part of the primitive is $-\frac{(a^2-x^2)^{\frac{3}{2}}}{3a}$. The constant part may be determined thus: When $x = o$, in the equation, $y = o$, and therefore the area $= o$. But in that case the above expression becomes $-\frac{a^2}{3}$; which, therefore, being subducted, we have the complete primitive, or area of the curve, $= \frac{a^2}{3} - \frac{(a^2-x^2)^{\frac{3}{2}}}{3a}$.

Again, when $x = a$, $y = o$, which shews that the curve meets the axis in two points; hence the area included by the curve and the axis must be $\frac{1}{3}a^2$.

EXAMPLE 6.

To find the area of the cycloid.

Let AD$=x$, DE$=y$, DF$=s$, the arc AF$(=$FE$)=z$, and AO$=$OB$=r$.



We have the ordinate $y = s + z$, of which the primitive is the area of the cycloidal space ADE. It is composed of the primitives of s and z. Admitting the former $\int s$, that is, the circular

area ADF, to be given, the latter fz, or the area AEF, may be found thus :

Since $rs' = \text{cos. } z \,.\, z' = (r - x)z'$, (D. 29.), therefore

$$rs' - rz' + xz' + z = z.$$

Consequently $rs - rz + xz = fz =$ area AEF ; which, when $x = r = s$, gives the cycloidal space APQ $= r^2$: And when $x = 2r$, s vanishes, and $fz = rz$ = APBCEA = the generating circle.

Hence the cycloid ABC = three times the semicircle APB.

EXAMPLE 7.

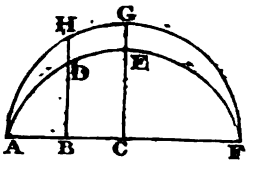
To find the area of an hyperbola.

From the equation $y^2 = \frac{b^2}{a^2}(2ax + x^2)$, putting $\overline{a + x} = z$, we get $yx' = \frac{bz'\sqrt{z^2 - a^2}}{a}$, of which the primitive is $\frac{bz\sqrt{z^2 - a^2}}{2a} - \frac{ab}{2} \,.\, l \,.\, (z + \sqrt{z^2 - a^2})$. But this expression, when $x = o$, (or $z = a$) becomes $- \frac{ab}{2} \, l \,.\, a$. The correct primitive, therefore, or the area contained by the curve, ordinate, and absciss, (which vanishes when $x = o$,) must be $\frac{bz\sqrt{z^2 - a^2}}{2a} - \frac{ab}{2} l. \frac{z + \sqrt{z^2 - a^2}}{a}$.

EXAMPLE 8.

To find the area of an ellipse.

From the equation, $y^2 = \frac{b^2}{a^2}(2ax - x^2)$, putting $z = \sqrt{a - x}$, we have $yx' = -\frac{bz\sqrt{a^2 - z^2}}{a}$, of which the primitive is $-\frac{b z \sqrt{a^2 - z^2}}{2a}$



$-\frac{ab}{2}$. arc $\left(\text{sin.} \frac{z}{a}\right)$. But this expression, when $x = o$, (or $z = a$,) becomes $-\frac{ab}{2}$.Q, (putting Q for the quadrant of a circle whose radius is 1). The correct primitive, therefore, or the elliptical area ADB, must be $\frac{ab}{2}(Q - A) - \frac{bz\sqrt{a^2 - z^2}}{2a}$, (where sin. $A = \frac{z}{a}$).

Hence, when $x = a$, z, and, consequently, A vanish; and we have $\frac{ab}{2}$. Q for $\frac{1}{4}$ of the ellipse; the whole ellipse, therefore, $= 2ab$. Q.

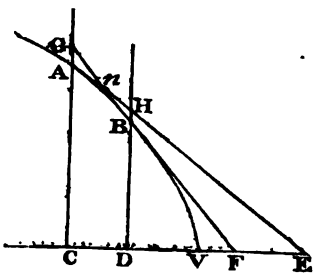
If the above expression be applied to the circle AGF, whose radius $= a$, we have $b = a$, the arc $A = \frac{GH}{a}$, and $\sqrt{Q - A} = \frac{AH}{a}$. Hence the first part of the ex-

pression becomes $\frac{1}{2}$ AC . AH = sector ACH; and the second becomes $\frac{1}{2}$ BC . BH = triangle CBH; the difference between which is the semi-segment AHB: And these parts of the circle are to the corresponding parts of the ellipse in the constant ratio of a to b.

SECTION III.

RECTIFICATION OF CURVE LINES, AND QUADRATURE OF CURVE SURFACES.

7. Assuming the principle, that any portion AB of a curve is greater than the straight line joining its extreme points, but less than the sum of the tangents An, Bn, drawn from these points to intersect each other, it follows, that the arc AB is greater than the tangent AnH, but less than the tangent BnG, intercepted by the ordinates AC, BD passing through A, B.



Now, if the abscis VD = x, and the ordinate

BD $= fx$, the tangent of the angle BFD will be expressed by $f'x$; and, if CD $= i$, the ordinate AC $= f(x + i)$, and tangent of AEC $= f'(x + i)$. Hence BG $= \sqrt{i^2 + (if'x)^2} = i\sqrt{1 + (f'x)^2}$, and AH $= i\sqrt{1 + (f' \cdot x + i)^2}$, which are the limits of the arc AB.

Suppose the former to be denoted by $i\mathrm{F}x$, then the latter will be expressed by $i\mathrm{F}(x + i)$; and, supposing φx to be the function of x, which expresses the length of the curve VB, then $\varphi(x + i) - \varphi x$ will express the length of the arc AB; which quantity, therefore, must be of an intermediate magnitude between $i\mathrm{F}(x+i)$ and $i\mathrm{F}x$. But this cannot be, unless $\varphi' x = \mathrm{F}x$. Considering $\mathrm{F}x$, or $\sqrt{1 + (f'x)^2}$, therefore, as a prime function, its primitive must express the length of the curve in terms of the absciss.

Thus, if z denote the length of the curve, and y, z, be considered as functions of x, we have,

$$z' = \sqrt{1 + y'^2}.$$

Or, if x, y, z are all to be regarded as functions of some other variable, the equation will be,

$$z' = \sqrt{x'^2 + y'^2}.$$

8. Again, if the curve VBA revolve about its axis

VC, it describes a curve surface: And it may be shewn, in like manner, that a plane surface equal to that curve surface may be found by taking the primitive of

$$cyz', \text{ or } cy\sqrt{x'^2 + y'^2},$$

c denoting the circumference of a circle whose radius is 1.

EXAMPLE 1.

Let the curve, whose length is required, be the common parabola.

The equation being $2ax = y^2$, we have $x'^2 = \frac{y^2y'^2}{a^2}$;

therefore $z' = \sqrt{\frac{y^2y'^2}{a^2} + y'^2} = \frac{1}{a} \cdot y'\sqrt{a^2 + y^2}$.

Hence $z = \frac{1}{2a} \cdot y\sqrt{a^2+y^2} + \frac{1}{2}a^2 \cdot l \cdot \frac{y+\sqrt{a^2+y^2}}{a}$.

EXAMPLE 2.

Required the length of the semi-cubical parabola, whose equation is $ax^2 = y^3$.

Here $2axx' = 3y^2y'$, and $x'^2 = \frac{9y^4y'^2}{4a^2x^2} = \frac{9y}{4a} \cdot y'^2$;

therefore $z' = y'\sqrt{\frac{9y}{4a} + 1}$, whereof the primitive is

$\frac{8a}{27} \cdot \left(\frac{9y}{4a} + 1\right)^{\frac{1}{2}}$, which, when $y = o$, becomes $\frac{8a}{27}$. Hence the correct primitive, or length of the curve, is

$$z = \frac{\overline{9y + 4a}^{\frac{3}{2}} - \overline{2a}^{\frac{3}{2}}}{27a^{\frac{1}{2}}}.$$

EXAMPLE 3.

To find the length of any portion of the cycloid.

Put $z =$ AE, (see Example 6. preceding Section); then because the tangent at E is parallel to the chord AF, $y' : x' = s : x$, and $y' = \frac{sx'}{x}$. Hence, (supposing $x' = 1$) $y'^2 = \frac{s^2}{x^2}$, and $z' = \sqrt{1 + y'^2} = \sqrt{1 + \frac{s^2}{x^2}}$ $= \frac{1}{x}\sqrt{s^2 + x^2} = \frac{1}{x}\sqrt{2rx} = \sqrt{\frac{2r}{x}}$. Consequently, $z = \sqrt{8rx} =$ double the chord AF of the generating circle.

Hence, half the cycloid, viz. AEC = twice the diameter AB.

EXAMPLE 4.

To find the curve surface of the segment of a sphere.

If a semicircle revolve about its diameter ($2r$), its circumference describes the surface of a sphere. Let

the arc z describe the surface of a segment; the abscis x being the altitude of the segment, and the ordinate y, the radius of its base. This curve surface is equal to the primitive function cyz', or of $cy\sqrt{1+y'^2}$, supposing $x'=1$. But, since $y^2=2rx-x^2$, $yy'=r-x$, and $y'^2=\frac{\overline{r-x}^2}{y}$. Hence $cyz'=c\sqrt{y^2+(r-x)^2}$ $=cr$; consequently, $\int cyz'=crx$; that is, the required surface is equal to the circumference of the generating circle multiplied by the altitude of the segment.

Hence, the whole surface of the sphere is equal to four times the area of a great circle.

EXAMPLE 5.

To find the curve surface of the parabolic conoid.

From the equation of the parabola, $y^2=2ax$, we get $y'^2=\frac{a^2}{y^2}$ (supposing $x'=1$). Hence $cyz'=cy\sqrt{1+\frac{y^2}{a^2}}$ $=c\sqrt{a^2+y^2}=c\sqrt{a^2+2ax}$; therefore, $\int cyz$ $=\frac{c(a^2+2ax)^{\frac{3}{2}}}{3a}$, which, when $x=o$, becomes $\frac{ca^2}{3}$; the complete primitive, therefore, or required surface, $=\frac{c(a^2+2ax)^{\frac{3}{2}}}{3a}-\frac{ca^2}{3}$.

SECTION IV.

RESOLUTION OF COMPOUND RATIONAL FRACTIONS.

9. LET $\frac{M}{N}$ denote any rational fraction, which is a function of x, and whose denominator N is resolvable into factors of the form x^n, $\overline{x+a}$, or x^2+a, it is proposed to shew how it may be resolved into two or more fractions.

Suppose $\frac{M}{N} = \frac{A}{x+a} + \frac{P}{Q}$, where P, Q are functions of x, and A some constant quantity.

Because $N = (x+a)\,Q$ by hypothesis, $M = AQ + (x+a).P$. Hence $M - AQ$ must be divisible by $\overline{x+a}$; that is, if $-a$ be substituted for x in $M - AQ$, the said quantity $M - AQ$ vanishes; if, therefore, by such a substitution, M becomes m, and Q becomes q, we shall have $m - Aq = o$, and $A = \frac{m}{q}$, the numerator of the required fraction.

Again, since $Q = \frac{N}{x+a}$, when $\overline{x+a} = o$, as

in that case $N = o$, we shall also have $Q = \frac{N'}{1} = N'$ (D. 44.). Thus we may find q, by substituting $-a$ for x in the prime function of N.

EXAMPLE 1.

Let $\frac{12x}{x^4 + x^3 - 2}$ be the proposed fraction, its denominator being composed of the factors $\overline{x+1}$ $\overline{x-1}$, and $x^2 + 2$.

Here $M = 12x$, which, when the factor $\overline{x+1} = o$, or $x = -1$, becomes $m = -12$; and $Q = N' = 4x^3 + 2x$, which, when $x = -1$, becomes $q = -6$.

Hence $A = \frac{-12}{-6} = 2$; and the first fraction $\frac{2}{x+1}$. In like manner, by putting the next factor $x - 1 = o$, or $x = 1$, we get the second fraction $\frac{2}{x-1}$. And, the sum of these two fractions being subtracted from the given one, the remainder, viz. $-\frac{4x}{x^2+1}$, is the third fraction.

Thus, $\frac{12x}{x^4 + x^3 - 2} = \frac{2}{x+1} + \frac{2}{x-1} - \frac{4x}{x^2+2}$.

And thus may any fraction be resolved into as many

component parts as there are factors in its denominator. For, when n factors are equal, the first, second, third, . . . n^{th} powers of that factor are to be taken as the denominators of the parts. If the fraction, for example, be

$\frac{1}{x^6+x^7-x^4-x^3}$, (where $N = x^3.\overline{x+1}^2.\overline{x-1}.\overline{x^2+1}$,) it may be resolved into seven fractions, whose denominators are, x, x^2, x^3, $\overline{x+1}$, $\overline{x+1}^2$, $\overline{x-1}$, and $\overline{x^2+1}$.

10. When the denominators are powers of one and the same quantity, the numerators are to be found thus:

Suppose $\frac{M}{N} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{P}{Q}$, then $N = Q.x^3$,

$$M = AQ + BQx + CQx^2 + Px^3,$$

and $$\frac{M}{Q} = A + Bx + Cx^2 + \frac{P}{Q}x^3.$$

Consequently, when $x = 0$, $A = \frac{M}{Q}$. And, by taking the derivative functions of $\frac{M}{Q}$ successively, we get

$$B = \frac{M}{Q}', \quad 2C = \frac{M}{Q}'', \quad 2.3.D = \frac{M}{Q}''', \text{ \&c.}$$

So that the numerators of the several fractions may be found independently of one another.

EXAMPLE 2.

Required the parts of the fraction, $\frac{1}{x^5+x^4-x^3-x^2}$, whose denominators are x^2, x^2, x; $\overline{x+1}\Big|^2$, $\overline{x+1}$.

Here $Q = x^5 + x^4 - x - 1 =$ (suppose) $z - 1$; then, when $x = o$, $z = o$, $z' = -1$, and $z'' = o$.

Hence $A = \frac{M}{Q} = \frac{1}{z-1} =$ (when $x = o$) $= -1$.

$$B = \frac{\overline{M}'}{Q}\Big| = -\frac{z'}{(z-1)^2} = -\frac{-1}{+1} = +1.$$

$$2C = \frac{\overline{M}''}{Q}\Big| = -\frac{z''}{(z-1)^2} + \frac{2z'z'}{(z-1)^3} = \frac{+2}{-1} = -2.$$

And $\frac{A}{x^2} = -\frac{1}{x^2}$, $\frac{B}{x^2} = +\frac{1}{x^2}$, and $\frac{C}{x} = -\frac{1}{x}$.

For the factors $\overline{x+1}\Big|^2$, $\overline{x+1}$, the numerators are found in the same manner, putting $x = -1$, instead of $x = o$, to be $+\frac{1}{4}$, and $+\frac{9}{8}$.

11. The above fraction, $\frac{1}{x^5+x^4-x^3-x^2}$, we have said, may be divided into seven fractions, whose denominators are $\overline{x-1}$, x^2, x^2, $\overline{x+1}\Big|^2$, $\overline{x+1}$ and $\overline{x^2+1}$; and we have shewn how to find the numerators for six of these fractions. The numerator for the remaining

one, (of which $\overline{x^2+1}$ is the denominator,) may also be obtained by the first process: Thus,

Since $Q = \frac{N}{x^2+1} = \frac{N'}{2x} = 4x^4 + \frac{7}{2}x^3 - 2x^2 - \frac{3}{2}x$,

which, when $x^2 + 1 = o$, or $x = \sqrt{-1}$, becomes $q = -2 + 2\sqrt{-1}$; therefore $A = \frac{m}{q} = \frac{1}{2\sqrt{-1}-2}$

$$= -\frac{2\sqrt{-1}+2}{8} - \tfrac{1}{4}(\sqrt{-1}+1) = -\tfrac{1}{4}(x+1).$$

So that, in this case, $-\frac{1}{4}(x+1)$, and not a constant quantity as before, must be the numerator of the required fraction.

The given fraction, therefore, $\frac{1}{x^6+x^7-x^4-x^3}$

$$= \frac{1}{8(x-1)} - \frac{1}{x^3} + \frac{1}{x^2} - \frac{1}{x} + \frac{1}{4(x+1)} + \frac{9}{8(x+1)} - \frac{x+1}{4(x^2+1)}.$$

12. Hence it appears, that if $\overline{x^2+a}$ be one of the factors, we must give it a numerator of the form $Ax + B$, which may be determined by this general process:

Since $\frac{M}{N} = \frac{Ax+B}{x^2+a} + \frac{P}{Q}$, and $N = Q \cdot (x^2 + a)$, $M - \overline{Ax+B} \cdot Q = P \cdot (x^2 + a)$, which, when

$\overline{x^2 + a} = o$, gives $Ax + B = \frac{M}{Q} = \frac{m}{q}$, (by substituting $-a$ for x^2 in M and Q); from which equation A and B are to be determined, by equating the constant terms thereof, and equating likewise the variable terms of the same.

Here, also, because $Q = \frac{N}{x^2 + a}$, (both of which terms vanish when $x^2 = -a$,) $Q = \frac{N'}{2x}$; so that, instead of $\frac{M}{Q}$, we may take $\frac{2Mx}{N'}$; from which, by substituting $-a$ for x^2, we shall get $\frac{m}{q}$.

EXAMPLE 3.

Let $\frac{x^3 - 2x^2 + 3x - 5}{x^4 - 2x^3 - 5x^2 + 4x + 6}$ be the proposed fraction, having $\overline{x^2 - 2}$ for one of the factors of its denominator.

We have $\frac{2Mx}{N'} = \frac{2x^4 - 4x^3 + 6x^2 - 10x}{4x^3 - 6x^2 - 10x + 4}$, in which, substituting 2 for x^2, it becomes $\frac{m}{q} = \frac{9x - 10}{x + 4}$ $= Ax + B$.

Hence $9x - 10 = 2A + 4Ax + Bx + 4B.$

Now, putting $2A + 4B = -10,$

we have $4A + B = 9.$

Whence $A = \frac{23}{7}$, and $B = -\frac{29}{7}$.

The required fraction, therefore, $\frac{Ax+B}{x^2-2} = \frac{23x-29}{7(x^2-2)}$.

For the factors $\overline{x-3}$ and $\overline{x+1}$, we get $\frac{13}{28(x-3)}$ and $-\frac{11}{4(x+1)}$; so that the given fraction

$$\frac{x^3-2x^2+3x-5}{x^4-2x^3-5x^2+4x+6} = \frac{23x-29}{7(x^2-2)} + \frac{13}{28(x-3)} - \frac{11}{4(x+1)}.$$

13. The symbol $\sqrt{-1}$, when it occurs in the solution of a problem, points out its impossibility. Thus, in Example 2, (11.), when we were seeking a *constant* quantity A, for the numerator of the fraction whose denominator was $x^2 + 1$, the result, involving $\sqrt{-1}$, shews that there is none to be found. Yet, by means of the same symbol, we were enabled to discover the variable quantity, viz. $-\frac{1}{4}(x + 1)$, which answers the conditions of the problem. And, in fact, however paradoxical it may appear, the sign of impossibility may be made subservient to the investigation of important truths, as will be shewn in the following Sections; but

we shall first give another view of the subject treated of in this.

14. Let $\frac{M}{N}$ be any rational fraction of the form,

$$\frac{a + bx + cx^2 + dx^3 + \&c.}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.}$$

of which the denominator N is a finite polynome of the degree n, (that is, x^n is the highest power of x in the said polynome,) and the numerator M is another of the inferior degree $\overline{n-1}$: Then, if N be the product of two polynomes or factors, R and Q, $\frac{M}{N}$ may be resolved into two rational fractions, $\frac{O}{R}$ and $\frac{P}{Q}$, of similar degrees.

For, if $\frac{M}{N} = \frac{O}{R} + \frac{P}{Q}$, and R be of the degree r, Q must be of the degree $\overline{n-r}$, since $N = RQ$; and if O and P be of the next inferior degree, $\overline{r-1}$ and $\overline{n-r-1}$, since $M = OQ + PR$, we shall have both sides of this equation of the same degree $\overline{n-1}$; and, therefore, the number of terms on each side is expressed by n. Hence, by equating the co-efficients of the same powers of x on each side, we shall have as

many equations as there are terms in O and P together; consequently these co-efficients may be found; that is, the numerators O and P may always be determined.

Suppose, then, the denominator $R = (\alpha - x)^r$, and $Q = A + A_1x + A_2x^2 + \ldots + A_{n-r}x^{n-r}$, and let it be required to find the fraction $\frac{O}{R}$.

Since $M = OQ + PR$, we have $O = \frac{M}{Q} - \frac{P}{Q}.(\alpha - x)^r$; which equation being true for all values of x, must be so when $\alpha - x = o$, or $x = \alpha$; but, in this case,

$$O = \frac{M}{Q} = \frac{a + bx + cx^2 + \ldots + a_{n-1}x^{n-1}}{A + A_1x + A_2x^2 + \ldots + A_{n-r}x^{n-r}} = \varphi x,$$

some function of x.

In other cases, when $\alpha - x = z$, we have still $\overline{\alpha - z} - x = o$, or $\overline{\alpha - z} = x$; therefore, considering α as variable, and z a decrement thereof, we have,

$$O = \varphi(\alpha - z) = \varphi\alpha - \varphi'\alpha.z + \tfrac{1}{2}\varphi''\alpha.z^2 - \frac{1}{2.3}\varphi'''\alpha \cdot z^3$$

$$\ldots\ldots + \frac{1}{1 \ldots r-1}\varphi^{r-1}\alpha \cdot z^{r-1}.$$

$$\frac{O}{R} = \frac{\varphi\alpha}{z^r} - \frac{\varphi'\alpha}{z^{r-1}} + \frac{\varphi''\alpha}{2.z^{r-2}} - \frac{\varphi'''\alpha}{2.3.z^{r-3}} + \ldots\ldots \pm \frac{\varphi^{r-1}\alpha}{1 \ldots r-1.z}$$

$$= \frac{\varphi x}{(\alpha - x)^r} - \frac{\varphi' x}{(\alpha - x)^{r-1}} + \frac{\varphi'' x}{2(\alpha - x)^{r-2}} - \frac{\varphi''' x}{2.3(\alpha - x)^{r-3}} + \&c.$$

provided we substitute a for x in φx, and its derivative functions.

Hence it appears that the fraction $\frac{O}{R} = \frac{o}{(a-x)^r}$ may itself be resolved into as many fractions as there are units in r, or factors in its denominator.

EXAMPLE.

Given $\frac{M}{N} = \frac{8-9x}{(2-x)^3 . x} = \frac{O}{(2-x)^3} + \frac{P}{x}$; required the fractions into which $\frac{O}{(2-x)^3}$ may be resolved.

Here $\varphi x = \frac{M}{Q} = \frac{8-9x}{x} = \frac{8}{x} = -9$, $\varphi' x = -\frac{8}{x^2}$, and $\frac{1}{2}\varphi'' x = \frac{8}{x^3}$, in each of which functions, substituting 2 for x, we get,

$$O = -5 + 2x + x^2,$$

$$\text{and } \frac{O}{R} = -\frac{5}{(2-x)^3} + \frac{2}{(2-x)^2} + \frac{1}{(2-x)}.$$

SECTION V.

ON THE USE OF THE SYMBOL $\sqrt{-1}$.

15. If x denote the distance of any point (within the curve) from the centre of a circle, or equilateral hyperbola, of which the radius, or semi-axis = 1, sin. z the corresponding ordinate in the former, and y that in the latter; then x being always less than 1 for the circle, and greater than 1 for the hyperbola, we have,

$$\text{sin.}^2 z = 1 - x^2,$$

$$\text{and} \qquad y^2 = x^2 - 1 = -\text{sin.}^2 z.$$

$$\text{Hence } y = \text{sin. } z\sqrt{-1}.$$

sin. $z\sqrt{-1}$, therefore, is an expression for the ordinate of the equilateral hyperbola in terms of the circle.

Now, because the sum of the squares of any sine and cosine, (as sin. z, cos. z,) is equal to the square of the radius = 1, and any power of 1 is 1, we have,

$$1 = (\text{cos. } z + \text{sin. } z\sqrt{-1}).(\text{cos. } z - \text{sin. } z\sqrt{-1}).$$

$$1 = (\text{cos. } z + \text{sin. } z\sqrt{-1})^n.(\text{cos. } z - \text{sin. } z\sqrt{-1})^n.$$

and $1 = (\text{cos. } nz + \text{sin. } nz\sqrt{-1}).(\text{cos. } nz - \text{sin. } nz\sqrt{-1}).$

The last two expressions for unity are not only equal themselves, but have their corresponding factors also equal, viz.

$$(\cos. z+\sin. z\sqrt{-1})^n = \cos. nz+\sin. nz\sqrt{-1},$$

and $(\cos. z-\sin. z\sqrt{-1})^n = \cos. nz-\sin. nz\sqrt{-1}.$

For the prime functions of their logarithms are equal, viz.

$$\frac{nz'(-\sin. z+\cos. z\sqrt{-1})}{\cos. z+\sin. z\sqrt{-1}},$$

and $$\frac{nz'(-\sin. nz+\cos. nz\sqrt{-1})}{\cos. nz+\sin. nz\sqrt{-1}};$$

because each of these quantities, multiplied by $-\frac{\sqrt{-1}}{nz'}$, becomes equal to unity; therefore,

$$l.(\cos. z+\sin. z\sqrt{-1})^n = l.(\cos. nz+\sin. nz\sqrt{-1},$$

no constant quantity being required, as both logarithms vanish when $z = o$. Consequently,

I. $(\cos. z+\sin. z\sqrt{-1})^n = \cos. nz+\sin. nz\sqrt{-1}.$

II. $(\cos. z-\sin. z\sqrt{-1})^n = \cos. nz-\sin. nz\sqrt{-1}.$

These are convenient formulæ for reducing the powers of sines to the signs of multiple arcs,—for expressing the sine of a multiple arc by the powers of the sine of the simple arc,—and for other purposes. For

it is plain that $\sqrt{-1}$ may be made to vanish, and yet z have a real value; thus, though sin. $z = o$, z itself may be any even multiple of a quadrant.

EXAMPLE 1.

To express any power of the cosine of an arc, in terms of the cosines of the multiple arcs.

Because

$$2\cos. z = \overline{\cos. z + \sin. z\sqrt{-1}} + \overline{\cos. z - \sin. z\sqrt{-1}} = a + b,$$

$$2^n \cos.^n z = a^n + na^{n-1}b + \frac{n \cdot n-1}{1 \cdot 2} a^{n-2}b^2$$

$$+ \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \&c.$$

$$\text{and } 2^n \cos.^n z = b^n + nb^{n-1}a + \frac{n \cdot n-1}{1 \cdot 2} b^{n-2}a^2$$

$$+ \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} b^{n-3}a^3 + \&c.$$

Hence, since $ab = 1$, we have, likewise,

$$2^n \cos.^n z = \frac{a^n + b^n}{2} + n\frac{a^{n-2} + b^{n-2}}{2} + \frac{n.n-1}{1.2} \cdot \frac{a^{n-4} + b^{n-4}}{2}$$

$$+ \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \cdot \frac{a^{n-6} + b^{n-6}}{2} + \&c.$$

But $a^n = \cos. nz + \sin. nz\sqrt{-1}.$

$b^n = \cos. nz - \sin. nz\sqrt{-1}.$

$$a^{n-2} = \cos.\overline{n-2}.z + \sin.\overline{n-2}.z\sqrt{-1}.$$
$$b^{n-2} = \cos.\overline{n-2}.z - \sin.\overline{n-2}.z\sqrt{-1}.$$

Therefore, $\frac{a^n + b^n}{2} = \cos. nz.$

$$\frac{a^{n-2} + b^{n-2}}{2} = \cos.\overline{n-2}.z, \text{ \&c.}$$

Consequently, carrying the series to half the number of terms in $(a + b)^n$,

$$2^{n-1}\cos.^n z = \cos. nz + n\cos.\overline{n-2}.z + \frac{n.n-1}{1.2}\cos.\overline{n-4}.z + \text{\&c.}$$

When n is an integral number, this series terminates. The number of terms in $(a+b)^n$ is expressed by $\overline{n+1}$, and therefore the number in the series must be $\frac{n+1}{2}$. Now $\overline{n+1}$ is divisible by 2, when n is an odd number, but when n is even, the half of the middle term is to be taken. In this latter case, the middle term of $(a+b)^n$ being $\frac{n.n-1.n-2\ldots(\frac{n}{2}+1)}{1.2.3\ldots.\frac{n}{2}} a^{\frac{n}{2}} b^{\frac{n}{2}}$, and $a^{\frac{n}{2}} b^{\frac{n}{2}} = 1$, half its co-efficient will be the concluding term of the series.

EXAMPLE 2.

To express the cosine of any multiple arc by the powers of the cosines of the simple arc.

Let $x = \cos. z + \sin. z\sqrt{-1}$.

Then $\frac{1}{x} = \cos. z - \sin. z\sqrt{-1}$.

Also $x^n = \cos. nz + \sin. nz\sqrt{-1}$.

And $\frac{1}{x^n} = \cos. zn - \sin. nz\sqrt{-1}$.

Hence, $x + \frac{1}{x} = 2\cos. z$. And $x^n + \frac{1}{x^n} = 2\cos. nz$.

Thus putting $2\cos. z = 6$, we have $x^2 - 6x + 1 = o$; from which equation, we find (D. 67), the sum of the n^{th} powers of its roots x and $\frac{1}{x}$ expressed as follows:

$$x^n + \frac{1}{x^n} = 6^n + \frac{n}{n-1}\left|\frac{6^{n-1}}{.}\right|' + \frac{n}{n-2}\left|\frac{6^{n-2}}{.}\right|'' + \frac{n}{n-3}\left|\frac{6^{n-3}}{.}\right|''' + \&c.$$

or, developing the polynomial functions, γ being $=-1$, $\delta = o$, $\epsilon = o$, &c.

$$= 6^n - \frac{n}{n-1}\cdot\overline{n-1}6^{n-2}\gamma + \frac{n}{n-2}\cdot\frac{n-2.n-3}{1.2}\cdot 6^{n-4}\gamma^2$$

$$- \frac{n}{n-3}\cdot\frac{n-3.n-4.n-5}{1.2.3}6^{n-6}\gamma^3 + \&c.$$

Consequently,

$$2\cos. nz = \mathfrak{C}^n - n\mathfrak{C}^{n-2} + \frac{n.n-3}{1.2}\mathfrak{C}^{n-4} - \frac{n.n-4.n-5}{1.2.3}\mathfrak{C}^{n-6}$$
$$+ \frac{n.n-5.n-6.n-7}{1.2.3.4}\mathfrak{C}^{n-8} - \&c.$$

where $\mathfrak{C} = 2\cos. z$.

16. Again, because the prime function of $l.(\cos. z + \sin. z\sqrt{-1})$ is $z' \cdot \frac{-\sin. z + \cos. z\sqrt{-1}}{\cos. z + \sin. z\sqrt{-1}} = z'\sqrt{-1}$, whose primitive is $z\sqrt{-1}$, (requiring no correction,) therefore,

I. Log. $(\cos. z + \sin. z\sqrt{-1}) = +z\sqrt{-1}$.

II. Log. $(\cos. z - \sin. z\sqrt{-1}) = -z\sqrt{-1}$.

17. Hence the following formulæ respecting the relations of logarithms and circular arcs, in which ϵ denotes the base of the hyperbolic logarithms.

I. $\epsilon^{z\sqrt{-1}} = \cos. z + \sin. z\sqrt{-1}$.

II. $\epsilon^{-z\sqrt{-1}} = \cos. z - \sin. z\sqrt{-1}$.

III. $\epsilon^{nz\sqrt{-1}} = \cos. nz + \sin. nz\sqrt{-1}$.

IV. $\epsilon^{-nz\sqrt{-1}} = \cos. nz - \sin. nz\sqrt{-1}$.

EXAMPLE 1.

$\frac{1}{2}z = \sin. z - \frac{1}{2}\sin. 2z + \frac{1}{3}\sin. 3z - \frac{1}{4}\sin. 4z + \&c.$

For, putting $v^{z\sqrt{-1}} = v$, we have $z\sqrt{-1} = l.v$, and,

By I. and II. (17.) . . . . $\frac{v - v^{-1}}{2} = \text{sin. } z\sqrt{-1}.$

By III. and IV. . . . . $\frac{v^n - v^{-n}}{2} = \text{sin. } nz\sqrt{-1}.$

But (D. 27.) $l.v = \frac{v - v^{-1}}{1} - \frac{v^2 - v^{-2}}{2} + \frac{v^3 - v^{-3}}{3} - \frac{v^4 - v^{-4}}{4} + \&c.$

Therefore, $\frac{1}{2} z\sqrt{-1} = \text{sin. } z\sqrt{-1} - \frac{1}{2} \text{sin. } 2z\sqrt{-1} + \frac{1}{3} \text{sin. } 3z\sqrt{-1} - \&c.$

And $\frac{1}{2}z = \text{sin. } z - \frac{1}{2}\text{sin. } 2z + \frac{1}{3}\text{sin. } 3z - \frac{1}{4}\text{sin. } 4z + \&c.$

Hence, by taking the prime functions of these, and dividing each side of the equation by z', we get the value of another series, viz. :

$\frac{1}{2} = \text{cos. } z - \text{cos. } 2z + \text{cos. } 3z - \text{cos. } 4z + \text{cos. } 5z - \&c.$

EXAMPLE 2.

To find (t) the true anomaly of a planet, (reckoned from the aphelion,) in terms of (x) the eccentric anomaly, and (c) the eccentricity, by means of the following equation :

$$\text{tan. } \tfrac{1}{2} t = \frac{\sqrt{1-c}}{\sqrt{1+c}} \cdot \text{tan. } \tfrac{1}{2} x.$$

We have then $\frac{\text{sin. } \frac{1}{2} t}{\text{cos. } \frac{1}{2} t} = \frac{\sqrt{1-c}}{\sqrt{1+c}} \cdot \frac{\text{sin. } \frac{1}{2} x}{\text{cos. } \frac{1}{2} x}.$

But $\cos.\frac{1}{2}t + \sin.\frac{1}{2}t\sqrt{-1} = \varepsilon^{\frac{1}{2}t\sqrt{-1}}$.

And $\cos.\frac{1}{2}t - \sin.\frac{1}{2}t\sqrt{-1} = \varepsilon^{-\frac{1}{2}t\sqrt{-1}}$.

Therefore $\cos.\frac{1}{2}t = \frac{\varepsilon^{\frac{1}{2}t\sqrt{-1}} + \varepsilon^{-\frac{1}{2}t\sqrt{-1}}}{2}$

and $\sin.\frac{1}{2}t = \frac{\varepsilon^{\frac{1}{2}t\sqrt{-1}} - \varepsilon^{-\frac{1}{2}t\sqrt{-1}}}{2\sqrt{-1}}$.

Hence, $\frac{\varepsilon^{\frac{1}{2}t\sqrt{-1}} - \varepsilon^{-\frac{1}{2}t\sqrt{-1}}}{\varepsilon^{\frac{1}{2}t\sqrt{-1}} + \varepsilon^{-\frac{1}{2}t\sqrt{-1}}} = \frac{\sqrt{1-c}}{\sqrt{1+c}} \cdot \frac{\varepsilon^{\frac{1}{2}x\sqrt{-1}} - \varepsilon^{-\frac{1}{2}x\sqrt{-1}}}{\varepsilon^{\frac{1}{2}x\sqrt{-1}} + \varepsilon^{-\frac{1}{2}x\sqrt{-1}}}$,

then multiplying the terms of the first fraction by $\varepsilon^{\frac{1}{2}t\sqrt{-1}}$, and those of the last by $\varepsilon^{\frac{1}{2}x\sqrt{-1}}$, we get,

$$\frac{\varepsilon^{t\sqrt{-1}} - 1}{\varepsilon^{t\sqrt{-1}} + 1} = \frac{\sqrt{1-c}}{\sqrt{1+c}} \cdot \frac{\varepsilon^{x\sqrt{-1}} - 1}{\varepsilon^{x\sqrt{-1}} + 1}.$$

(These quantities being represented thus, $\frac{a-1}{a+1} = \frac{n}{m} \cdot \frac{b-1}{b+1}$, it is easy to find $a = \frac{\sqrt{m+n}.b + \sqrt{m-n}}{\sqrt{m+n} + \sqrt{m-n}.b}$

$$= \frac{b + \frac{m-n}{m+n}}{1 + \frac{m-n}{m+n} b}.)$$

Therefore, putting $\lambda = \frac{m-n}{m+n} = \frac{\sqrt{1+c} - \sqrt{1-c}}{\sqrt{1+c} + \sqrt{1-c}}$

$$= \frac{1 - \sqrt{1-c^2}}{c}, \quad \varepsilon^{t\sqrt{-1}} = \frac{\varepsilon^{x\sqrt{-1}} + \lambda}{1 + \lambda\varepsilon^{x\sqrt{-1}}}$$

$$= \varepsilon^{x\sqrt{-1}} \cdot \frac{1 + \lambda\varepsilon^{-x\sqrt{-1}}}{1 + \lambda\varepsilon^{x\sqrt{-1}}}.$$

In the former equation, instead of z, we may substitute $\overline{c+z}$, $\overline{2c+z}$, $\overline{3c+z}$, &c. (c denoting the whole circumference,) because all these arcs have the very same sine and cosine.

In the latter equation, therefore, the arc may be expressed by any term of the following series,

$$\frac{z}{n}, \frac{c+z}{n}, \frac{2c+z}{n}, \frac{3c+z}{n} \ldots\ldots \frac{\overline{n-1}.c+z}{n},$$

where the terms, from $\frac{z}{n}$ to $\frac{\overline{n-1}.c+z}{n}$ inclusive, have all different sines and cosines; but the terms which follow, viz. $\frac{nc+z}{n}$, $\frac{\overline{n+1}.c+z}{n}$, $\frac{\overline{n+2}.c+z}{n}$, &c. ($=c+\frac{z}{n}$, $c+\frac{c+z}{n}$, $c+\frac{2c+z}{2}$, &c.) have exactly the same sines and cosines with those which precede.

Thus it appears that x has as many different values as there are units in n; or, which is the same, that

$$x^n - r^n.\overline{\cos.z+\sin.z\sqrt{-1}} = (x-r.\cos.\frac{z}{n}+\sin.\frac{z}{n}\sqrt{-1}).$$

$$(x-r.\cos.\frac{c+z}{n}+\sin.\frac{c+z}{n}\sqrt{-1}).(x-r.\cos.\frac{2c+z}{n}$$

$$+\sin.\frac{2c+z}{n}\sqrt{-1}).\ \&c.$$

In like manner, $x^n - r^n.\overline{\cos.z-\sin.z\sqrt{-1}}$

$$=(x-r.\cos.\frac{z}{n}-\sin.\frac{z}{n}\sqrt{-1}).(x-r.\cos.\frac{c+z}{n}-\sin.\frac{c+z}{n}\sqrt{-1}).$$

$$(x-r.\cos.\frac{2c+z}{n}-\sin.\frac{2c+z}{n}\sqrt{-1}),\ \&c.$$

Consequently, by multiplying these equations, we get,

$$x^{2n}-2r^nx^n\cos.z+r^{2n}=(x^2-2rx\cos.\frac{z}{n}+r^2).$$

$$(x^2-2rx\cos.\frac{c+z}{n}+r^2),\ \&c.$$

continued to n factors.

Every trinomial, then, of the form $x^{2n}-2r^nx^n\cos.z+r^{2n}$, may be resolved into n trinomial factors of the form $x^2-2rx\cos.\frac{mc+z}{n}+r^2$; where m is equal to o, or to any number less than n.

19. From the last equation, by putting $z=\pi$, (half the circumference,) we have $\cos.z=-1$; next putting $z=2\pi$, we have $\cos.z=+1$; then extracting the square root on both sides, we get,

$$x^n+r^n=\sqrt{x^2-2rx\cos.\frac{\pi}{n}+r^2}.\sqrt{x^2-2rx\cos.\frac{3\pi}{n}+r^2}.$$

$$.\sqrt{x^2-2rx\cos.\frac{5\pi}{n}+r^2}.\ \&c.$$

$$x^n-r^n=\sqrt{x^2-2rx\cos.\frac{2\pi}{n}+r^2}.\sqrt{x^2-2rx\cos.\frac{4\pi}{n}+r^2}.$$

$$.\sqrt{x^2-2rx\cos.\frac{6\pi}{n}+r^2}.\ \&c.$$

The arcs in these factors compose the two following series:

$$\frac{\pi}{n}, \frac{3\pi}{n}, \frac{5\pi}{n}, \frac{7\pi}{n} \ldots\ldots\ldots \frac{\overline{2n-1}.\pi}{n}.$$

$$\frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \frac{8\pi}{n} \ldots\ldots\ldots \frac{2n\pi}{n}.$$

In the former series, it is evident that the sum of the first and last terms, and of every two terms equally distant from them, is 2π. But two arcs, whose sum is 2π, have exactly the same cosine. The first, second, third, &c. factors, therefore, will be respectively equal to the last, last but one, last but two, &c.: And if the number n be odd, the remaining factor will be

$$\sqrt{x^2 - 2rx \cos. \frac{n\pi}{n} + r^2} = \overline{x + r}.$$

In the latter series, the last term being $= 2\pi$, whose cosine is $+ 1$, the last factor will be $x - r$. Of the remaining factors, the first and last, the second and last but one, &c. will be equal, because their arcs have the very same cosine: And, if the number n be even, there will be an odd number of the remaining factors, whereof the middle one, involving the cosine of $\frac{n\pi}{n}$, will be, as in the preceding case, $x + r$. Hence,

$$x^n + r^n = (x^2 - 2rx\cos.\frac{\pi}{n} + r^2).(x^2 - 2rx\cos.\frac{3\pi}{n} + r^2).$$

$$.(x^2 - 2rx\cos.\frac{5\pi}{n} + r^2).\ \&c.$$

. . . . . to $\frac{1}{2}n$ factors, if n be even.

. . . . . to $\frac{n-1}{2}$ factors $\times(x+r)$, if n be odd.

$$x^n - r^n = (x^2 - 2rx\cos.\frac{2\pi}{n} + r^2).(x^2 - 2rx\cos.\frac{4\pi}{n} + r^2).$$

$$.(x^2 - 2rx\cos.\frac{6\pi}{n} + r^2).\ \&c.$$

. . . . . to $\left(\frac{n}{2}-1\right)$ factors $\times\ (x-r).(x+r)$, if n be even.

. . . . . to $\left(\frac{n-1}{2}\right)$ factors $\times\ (x-r)$, if n be odd.

20. Having resolved the binomial $x^n \pm a^n$ into factors, we may also resolve the fraction $\frac{x}{x^n \pm a^n}$ into as many parts as there are factors in its denominator. Thus:

Let the numerator of one of the fractions be $Ax+B$, so that $\frac{M}{N} = \frac{Ax+B}{x^2-2ax\cos.\varphi+a^2} + \frac{P}{Q}$, where the arc φ is a multiple of $\frac{\pi}{n}$ by an odd number for x^n+a^n, or by an even number for $x^n - a^n$: We have, (as in Sect. IV.)

$Ax + B = \frac{m}{q}$; but $Q = \frac{N}{x^2 - 2ax\cos.\varphi + a^2} = \frac{N'}{2x - 2a\cos.\varphi}$, hence, (putting $r = n - m - 1$), $\frac{M}{Q} = \frac{x^m(2x - 2a\cos.\varphi)}{N'}$ $= \frac{x\ (2x - 2a\cos.\varphi)}{nx^{n-1}} = \frac{2x - 2a\cos.\varphi}{nx^r}$. And $Ax + B$ $= \frac{m}{q}$ = (by substituting $a \,.\, \overline{\cos.\varphi + \sin.\varphi\sqrt{-1}}$ for x in the preceding quantity) $= \frac{2a\sin.\varphi\sqrt{-1}}{na^r(\cos.r\varphi + \sin.r\varphi\sqrt{-1})}$ $= \frac{2a\sin.\varphi\sqrt{-1}\ (\cos.r\varphi - \sin.r\varphi\sqrt{-1})}{na^r}$.

Thus $A.a(\cos.\varphi + \sin.\varphi\sqrt{-1}) + B = \frac{2a\sin.\varphi\sqrt{-1}.\cos.r\varphi}{na^r}$ $+ \frac{2a.\sin.\varphi\sin.r\varphi}{na^r}$. To get quit of $\sqrt{-1}$, we may put $A \,.\, a\sin.\varphi\sqrt{-1} = \frac{2a\sin.\varphi\sqrt{-1}.\cos.r\varphi}{na^r}$; from which $A = \frac{2\cos.r\varphi}{na^r}$; then also $A \,.\, a\cos.\varphi + B$ $= \frac{2a.\sin.\varphi.\sin.r\varphi}{na^r}$; from which $B = \frac{2a.\sin.\varphi.\sin.r\varphi}{na^r}$ $- \frac{2a\,.\cos.\varphi.\cos.r\varphi}{na^r} = \frac{2a\cos.\overline{r+1}\,.\varphi}{na^r}$.

Consequently, $Ax + B = \frac{2x\cos.\overline{n-m-1}.\varphi - 2a.\cos.\overline{n-m}.\varphi}{na^{n-m-1}}$.

SECTION VII.

OF SERIES WHICH PROCEED ACCORDING TO THE SINES OR COSINES OF THE MULTIPLES OF ONE AND THE SAME ARC.

THE utility of the preceding formulæ will be farther illustrated in the solution of the following general Problem.

To convert the following function,
$\varphi(\alpha + \beta \cos. z + \gamma \cos.^2 z + \delta \cos.^3 z + \varepsilon \cos.^4 z + \&c.)$
into a series of this form,
$A + B.\cos. z + C.\cos. 2z + D.\cos. 3z + E.\cos. 4z + \&c.$

The given function developed (D. 57.) becomes,

$$\varphi\alpha + \left|\frac{\varphi\alpha}{.}\right|'.\cos. z + \left|\frac{\varphi\alpha}{.}\right|''.\cos.^2 z + \left|\frac{\varphi\alpha}{.}\right|'''.\cos.^3 z + \&c.$$

Put $t = \cos. z + \sin. z\sqrt{-1}$, and $v = \cos. z - \sin. z\sqrt{-1}$; then $\cos. z = \frac{1}{2}(t + v)$, and $\cos. nz = \frac{1}{2}(t^n + v^n)$. Now, substituting $\frac{1}{2}(t + v)$, and its powers, in place of the powers of $\cos. z$, we have,

$$\varphi\alpha+\tfrac{1}{2}\left|\frac{\varphi\alpha}{\cdot}\right|'\cdot\begin{vmatrix} t+\tfrac{1}{4}\left|\frac{\varphi\alpha}{\cdot}\right|''\cdot \\ +v \end{vmatrix}\cdot\begin{vmatrix} t^2+\tfrac{1}{8}\left|\frac{\varphi\alpha}{\cdot}\right|'''\cdot \\ +2tv \\ +v^2 \end{vmatrix}\cdot\begin{vmatrix} t^3+\tfrac{1}{16}\left|\frac{\varphi\alpha}{\cdot}\right|^{IV}\cdot \\ +3t^2v \\ +3tv^2 \\ +v^3 \end{vmatrix}\cdot\begin{vmatrix} t^4+\&c. \\ +4t^3v \\ +6t^2v^2 \\ +4tv^3 \\ +v^4 \end{vmatrix}$$

Observe,

1. Each term is composed of as many parts as there are units in the number (n) denoting the place of that term in the series; and its co-efficient is $\frac{1}{2^{n-1}}$.

2. The middle part of every odd term is a multiple of some power of tv; and, since $tv = 1$, its general expression is $\frac{n-1\,.\,n-2\,.\,n-3\,.....\,\frac{n+1}{2}}{2^{n-1}\,.\,1\,.\,2\,.\,3\,.\,4.....\,\frac{n-1}{2}}\left|\frac{\varphi\alpha}{\cdot}\right|^{n-1}$.

3. With respect to the other parts of the n^{th} term, the sum of the first and last is $t^{n-1}+v^{n-1}(=\frac{1}{2}\cos.\overline{n-1}.z)$; of the second and last but one, $\overline{n-1}\,(t^{n-2}+v^{n-2}) = \frac{1}{2}\overline{n-1}\cos.\overline{n-2}\,.\,z$; of the third and last but two, $\frac{n-1\,.\,n-2}{1\,.\,2}(t^{n-3}+v^{n-3})$, &c.; because $tv = 1$.

4. The odd powers of t, v are found in the even terms, and the even powers in the odd terms of the series.

Hence, beginning with the middle parts of the odd terms, the preceding developement may be changed as follows:

$$\varphi\alpha+\tfrac{1}{4}\cdot\tfrac{2}{1}\left|\frac{\varphi\alpha}{.}\right|''+\frac{1}{16}\cdot\frac{4.3}{1.2}\left|\frac{\varphi\alpha}{.}\right|^{IV}+\frac{1}{64}\cdot\frac{6.5.4}{1.2.3}\left|\frac{\varphi\alpha}{.}\right|^{VI}+\&c.$$

$$+\left(\left|\frac{\varphi\alpha}{.}\right|'+\tfrac{1}{4}\cdot\tfrac{3}{1}\left|\frac{\varphi\alpha}{.}\right|'''+\frac{1}{16}\cdot\frac{5.4}{1.2}\left|\frac{\varphi\alpha}{.}\right|^{V}+\frac{1}{64}\cdot\frac{7.6.5}{1.2.3}\left|\frac{\varphi\alpha}{.}\right|^{VII}+\&c.\right)\cos.\ z$$

$$+\left(\tfrac{1}{2}\left|\frac{\varphi\alpha}{.}\right|''+\tfrac{1}{8}\cdot\tfrac{4}{1}\left|\frac{\varphi\alpha}{.}\right|^{IV}+\frac{1}{32}\cdot\frac{6.5}{1.2}\left|\frac{\varphi\alpha}{.}\right|^{VI}+\frac{1}{128}\cdot\frac{8.7.6}{1.2.3}\left|\frac{\varphi\alpha}{.}\right|^{VIII}+\&c.\right)\cos.\ 2z$$

$$+\left(\tfrac{1}{4}\left|\frac{\varphi\alpha}{.}\right|'''+\tfrac{1}{16}\cdot\tfrac{6}{1}\left|\frac{\varphi\alpha}{.}\right|^{V}+\frac{1}{64}\cdot\frac{7.6}{1.2}\left|\frac{\varphi\alpha}{.}\right|^{VII}+\frac{1}{256}\cdot\frac{9.8.7}{1.2.3}\left|\frac{\varphi\alpha}{.}\right|^{IX}+\&c.\right)\cos.\ 3z$$

$$+\ \&c. \qquad\qquad \&c.$$

The law of continuation is manifest.

22. Cor. Hence, if x be the complement of z in the preceding function, we shall also have,

$$\varphi(\alpha+\mathfrak{C}\sin.\ x+\gamma\sin.^2 x+\delta\sin.^3 x+\varepsilon\sin.^4 x+\&c.)$$

$$=\varphi\alpha+\tfrac{1}{4}\cdot\tfrac{2}{1}\left|\frac{\varphi\alpha}{.}\right|''+\frac{1}{16}\cdot\frac{4.3}{1.2}\left|\frac{\varphi\alpha}{.}\right|^{IV}+\frac{1}{64}\cdot\frac{6.5.4}{1.2.3}\left|\frac{\varphi\alpha}{.}\right|^{VI}+\&c.$$

$$+\left(\left|\frac{\varphi\alpha}{.}\right|'+\tfrac{1}{4}\cdot\tfrac{3}{1}\left|\frac{\varphi\alpha}{.}\right|'''+\frac{1}{16}\cdot\frac{5.4}{1.2}\left|\frac{\varphi\alpha}{.}\right|^{V}+\frac{1}{64}\cdot\frac{7.6.5}{1.2.3}\left|\frac{\varphi\alpha}{.}\right|^{VII}+\&c.\right)\sin.\ x$$

$$-\left(\tfrac{1}{2}\left|\frac{\varphi\alpha}{.}\right|''+\tfrac{1}{8}\cdot\tfrac{4}{1}\left|\frac{\varphi\alpha}{.}\right|^{IV}+\frac{1}{32}\cdot\frac{6.5}{1.2}\left|\frac{\varphi\alpha}{.}\right|^{VI}+\frac{1}{128}\cdot\frac{8.7.6}{1.2.3}\left|\frac{\varphi\alpha}{.}\right|^{VIII}+\&c.\right)\cos.\ 2x$$

$$-\left(\tfrac{1}{4}\left|\frac{\varphi\alpha}{.}\right|'''+\tfrac{1}{16}\cdot\tfrac{6}{1}\left|\frac{\varphi\alpha}{.}\right|^{V}+\frac{1}{64}\cdot\frac{7.6}{1.2}\left|\frac{\varphi\alpha}{.}\right|^{VII}+\frac{1}{256}\cdot\frac{9.8.7}{1.2.3}\left|\frac{\varphi\alpha}{.}\right|^{IX}+\&c.\right)\sin.\ 3x$$

$$+\ \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \cos.\ 4x$$

where the sine and cosine of the multiple arcs follow each other alternately; and the signs (+ and —) of the terms also follow alternately in pairs.

EXAMPLE.

To develope $(1 - \mathfrak{C}\cos. z)^{-z}$.

Here $\alpha = 1$, $\varphi\alpha = \alpha^{-z}$; and, since $\gamma = o$, $\delta = o$, &c. in the developement of the polynomial functions of $\varphi\alpha$, (Table II.), there remain only the powers of $-\mathfrak{C}$ to be multiplied by the derivatives of α^{-z}.

Thus, $\left|\frac{\varphi\alpha}{.}\right|' = -2\alpha^{-3}.-\mathfrak{C} = 2\mathfrak{C}.$

$\left|\frac{\varphi\alpha}{.}\right|'' = +\frac{2.3}{1.2}\alpha^{-4}.+\mathfrak{C}^2 = 3\mathfrak{C}^2.$

$\left|\frac{\varphi\alpha}{.}\right|''' = -\frac{2.3.4}{1.2.3}\alpha^{-5}.-\mathfrak{C}^3 = 4\mathfrak{C}^3.$

$\left|\frac{\varphi\alpha}{.}\right|^{IV} = +\frac{2.3.4.5}{1.2.3.4}\alpha^{-6}.+\mathfrak{C}^4 = 5\mathfrak{C}^4.$

&c. &c.

Therefore $(1 - \mathfrak{C}\cos. z)^{-z} =$

$$1 + \frac{2.3}{1.4}\mathfrak{C}^2 + \frac{3.4.5}{1.2.16}\mathfrak{C}^4 + \frac{4.5.6.7}{1.2.3.64}\mathfrak{C}^6 + \frac{5.6.7.8.9}{1.2.3.4.256}\mathfrak{C}^8 + \&c.$$

$$+\left(\frac{2}{1}\mathfrak{C} + \frac{3.4}{1.4}\mathfrak{C}^3 + \frac{4.5.6}{1.2.16}\mathfrak{C}^5 + \frac{5.6.7.8}{1.2.3.64}\mathfrak{C}^7 + \&c.\right).\cos. z$$

$$+\left(\frac{3}{2}\mathfrak{C}^2 + \frac{4.5}{1.8}\mathfrak{C}^4 + \frac{5.6.7}{1.2.32}\mathfrak{C}^6 + \frac{6.7.8.9}{1.2.3.128}\mathfrak{C}^8 + \&c.\right).\cos. 2z$$

$$+\left(\frac{4}{4}\mathfrak{C}^3 + \frac{5.6}{1.16}\mathfrak{C}^5 + \frac{6.7.8}{1.2.64}\mathfrak{C}^7 + \frac{7.8.9.10}{1.2.3.256}\mathfrak{C}^9 + \&c.\right).\cos. 3z$$

&c. &c.

The general term, (exclusive of the first,) putting n for the multiple of z, is

$$\frac{n+1}{2^{n-1}}\zeta^n + \frac{n+2.n+3}{1.2^{n+1}}\zeta^{n+2} + \frac{n+3.n+4.n+5}{1.2.2^{n+3}}\zeta^{n+4} + \frac{n+4\ldots n+7}{1.2.3.2^{n+5}}\zeta^{n+6} + \&c.;$$

(the half of which, when $n = o$, is equivalent to the first term). Now, this general term being divided by the first term, the quotient is

$$\frac{n+1}{2^{n-1}}\zeta^n + \frac{n.n-1}{1.2^{n+1}}\zeta^{n+2} + \frac{n.n-1.n+1}{1.2.2^{n+3}}\zeta^{n+4} + \frac{n.n-1.n+1.n+4}{1.2.3.2^{n+5}}\zeta^{n+6} + \frac{n.n-1.n+1.n+5.n+6}{1\ldots4.2^{n+7}}\zeta^{n+8} + \frac{n.n-1.n+1.n+6.n+7.n+8}{1\ldots.5.2^{n+9}}\zeta^{n+10} + \&c.$$

But the first term $=(1-\zeta^2)^{-\frac{1}{2}} = \frac{1}{(1-\zeta^2)^{\frac{1}{2}}} = \frac{1}{s}$ suppose;

then $(1-\zeta\cos.z)^{-\frac{1}{2}}$

$$= \frac{1}{s} + \frac{1}{s}\left(\frac{n+1}{2^{n-1}}\zeta^n + \frac{n.n-1}{1.2^{n+1}}\zeta^{n+2} + \frac{n.n-1.n+1}{1.2.2^{n+3}}\zeta^{n+4} + \&c.\right)\cos.nz;$$

where n is to be put $= 1, 2, 3,$ &c. successively.

SECTION VIII.

OF THE INTEGRABLE FUNCTIONS OF A SINGLE VARIABLE.

We propose now to resume the subject of Integration, and to explain some of the various methods used by ingenious analysts for discovering the primitive of a given prime function; or, when that is impracticable, of approximating its value.

We are first to give examples of integration in the more common forms of the integrable functions of a single variable; next, to point out a general method of investigating the primitive in certain forms of functions; and, lastly, to give the method of approximation by infinite series.

I. FORM, $\int (Ax^m + Bx^n + \&c.)\, x'$.

EXAMPLE 1.

If $y' = ax^m x'$, then $y = \frac{ax^{m+1}}{m+1} + c$.

Here, in order to determine the constant quantity c,

suppose (from the nature of the problem) that $x = b$, when $y = o$; then $c = -\frac{ab^{m+1}}{m+1}$, and $y = \frac{a(x^{m+1}-b^{m+1})}{m+1}$.

In this example, it is to be observed, that, in the particular case of $m = -1$, $y' = \frac{ax'}{x}$, and $y = al.x + c$.

II. FORM, $\int \left(\frac{Ax^m + Bx^{m-1} + \&c.}{ax^{m-1} + bx^{m-2} + \&c.} \right) x'$.

EXAMPLE 2.

Given $y' = \frac{Ax^2x'}{ax+b}$, to find the primitive y.

Put $z = ax + b$, then $x' = ax'$, and $y' = \frac{A(z-b)^2.x'}{a^2z}$ $= \frac{A}{a^3}(zz' - 2bz' + b^2z^{-1}.z')$; therefore $y = \frac{A}{a^3}(\frac{1}{2}z^2 - 2bz + b^2.l.z) + c$, or $y = \frac{A}{a^3}(\frac{1}{2}(ax+b).(ax-3b) + b^2.l.(ax+b)) + c$.

EXAMPLE 3.

Given $y' = \frac{Ax^mx'}{(ax+b)^n}$. Putting $z = \overline{ax+b}$, we get $y' = \frac{A(z-b)^m.z'}{a^{m+1}.z^n}$, which, ($m$ being any whole number,) may be resolved as in the preceding example.

Examples of this form may sometimes be reduced to those of the next: Thus,

EXAMPLE 4.

$$\text{Given } y' = \frac{(x^3+3x^2-1).x'}{x^2+x-2} = \left(x+2+\frac{3}{x^2+x-2}\right)x'.$$

Whence $y = \frac{1}{2}x^2+2x+\int\frac{3}{x^2+x-2}$. See example 12.

III. FORM, $\int\left(\frac{Ax^{m-1}+Bx^{m-2}+\&c.}{ax^m+bx^{m-1}+\&c.}\right)x'$.

EXAMPLE 5.

$$\text{Given } y' = \frac{Ax'}{ax\pm b}, \text{ then } y = \frac{A}{a}\,l.\,(ax\pm b).$$

EXAMPLE 6.

$$\text{Given } y' = \frac{Bx'}{ax^2+b}. \quad \text{Put } c^2 = \frac{b}{a}, \text{ then } y' = \frac{B}{ac}\cdot\frac{cx'}{x^2+c^2}.$$

$$\text{Hence } y = \frac{B}{ac}\,.\,\text{arc}\left(\tan.\frac{x}{c}\right). \quad (4.\text{ example } 1.)$$

EXAMPLE 7.

$$\text{Given } y' = \frac{Bx'}{ax^2-b}. \quad \text{Put } c^2 = \frac{b}{a}, \text{ then } y' = \frac{B}{2ac}\cdot\frac{2cx'}{x^2-c^2}.$$

$$\text{Hence } y = -\frac{B}{2ac}\,.\,l.\,\frac{x+c}{x-c} = \frac{B}{2ac}\,.\,l.\,\frac{x-c}{x+c}. \quad (3.).$$

EXAMPLE 8.

Given $y' = \frac{Axx'}{ax^2 \pm b}$, then $y = \frac{A}{2a} . l . (ax^2 \pm b)$.

EXAMPLE 9.

Given $y' = \frac{Axx' \pm Bx'}{ax^2 \pm b}$. This may be divided into two parts, of which the primitives are to be found in the two preceding examples.

EXAMPLE 10.

Given $y' = \frac{Ax'}{(a \pm bx)^n}$. Put $z = a \pm bx$, then $z' = \pm bx'$, and $x' = \pm \frac{z'}{b}$. Hence $y' = \pm \frac{Az'}{bz^n}$, and $y = \pm \frac{A}{(1-n)b} \cdot (a \pm bx)^{1-n}$.

EXAMPLE 11.

Given $y' = \frac{Axx'}{(a^2 \pm x^2)^n}$. Put $z = a^2 \pm x^2$, then, as above, $y' = \pm \frac{\frac{1}{2}Az'}{z^n}$; therefore $y = \pm \frac{A}{2(1-n)} . (a^2 \pm x^2)^{1-n}$.

When the denominator is resolvable into factors, the fraction may be divided into its component parts, by

the methods explained in the preceding Sections; and thence the several parts of the primitive may be more easily determined.

EXAMPLE 12.

Given $y' = \frac{3x'}{x^2+x-2}$. Here $x^2+x-2=(x-1).(x+2)$.

Hence $y' = \frac{x'}{x-1} - \frac{x'}{x+2}$, and $y = l.\frac{x-1}{x+2}$. See example 4.

EXAMPLE 13.

Given $y' = \frac{(x^3-2x^2+3x-5).x'}{x^4-2x^3-5x^2+4x+6}$. The denominator here $= (x^2-2).(x-3).(x+1)$; hence we get, (12. example 3.)

$$y' = \frac{(23x-29)x'}{7(x^2-2)} + \frac{13x'}{28(x-3)} - \frac{11x'}{4(x+1)};$$

and, (by examples 8, 7, 5, preceding,)

$$y = \tfrac{23}{14} l.(x^2-2) + \frac{29\sqrt{2}}{28} l.\frac{x+\sqrt{2}}{x-\sqrt{2}} + \tfrac{13}{28} l.(x-3) - \tfrac{11}{4} l.(x+1).$$

EXAMPLE 14.

Given $y' = \frac{x'}{x^5+x^4-x^3-x^2} = \frac{x'}{8(x-1)} + \frac{x'}{4(x+1)^2} + \frac{9x'}{8(x+1)} - \frac{x'}{x^2} + \frac{x'}{x^3} - \frac{x'}{x} - \frac{(x+1)x'}{4(x^2+1)}$, (11.).

Hence we get

$$y = \tfrac{1}{8} l.(x-1) - \frac{1}{4(x+1)} + \tfrac{9}{8} l.(x+1) + \frac{1}{2x^2}$$

$$-\frac{1}{x} - l.x - \tfrac{1}{8} l.(x^2+1) - \tfrac{1}{4} \text{ar. (tan. } x);$$

$$\text{or, } y = l.\frac{x+1}{x} + \tfrac{1}{8} l.\frac{x^2-1}{x^2+1} - \frac{5x^3+2x-2}{4x^2(x+1)}$$

$$-\tfrac{1}{4} \text{ ar. (tan. } = x). \quad \text{See example 6.}$$

EXAMPLE 15.

Given $y' = \frac{x^m x'}{px^n \smile q}$. By putting $\frac{q}{p} = a^n$, the equation becomes $y' = \frac{1}{p} \cdot \frac{x^m x'}{x^n \smile a^n}$. But (19.)

$$x^n \smile a^n = (x^2 - 2ax \cos. \tfrac{c}{n} + a^2).(x^2 - 2ax \cos. \tfrac{2c}{n} + a^2)$$

$$.(x^2 - 2ax \cos. \frac{3c}{n} + a^2) \text{ &c.}$$

Let φ denote one of these arcs, $\left(\frac{c}{n}, \frac{2c}{n}, \frac{3c}{n}, \text{&c. } c = 360^\circ,\right)$ then the corresponding partial fraction of $\frac{x^m x'}{x^n \smile a^n}$ will be expressed by $\frac{2 \cos. r\varphi x x' - 2a \cos. \overline{r+1}.\varphi.x'}{na^r(x^2 - 2ax \cos. \varphi + a^2)}$, where $r = n - m - 1$, (20.). Hence, putting $z = x - a \cos. \varphi$, we shall have $z' = x'$, and the said fraction

$$= \frac{\cos. r\varphi \,.\, zz' + a \cos. \varphi \,.\, \cos. r\varphi z' - a \cos. \overline{r+1} \,.\, \varphi \,.\, z'}{\frac{1}{2} na^r \,.\, (z^2 + a^2 \sin.^2 \varphi)}$$

$$= \frac{\cos. r\varphi zz' + a \sin. \varphi \,.\, \sin. r\varphi z'}{\frac{1}{2} na^r (z^2 + a^2 \sin.^2 \varphi)}, \text{ of which the primi-}$$

tive (see example 9.) is $\frac{1}{\frac{1}{2} na^r} \left(\frac{1}{2} \cos. r\varphi . l . (z^2 + a^2 \sin.^2 \varphi) \right.$

$\left. + \sin. r\varphi \,.\, \text{arc (tan.} = \frac{z}{a \sin. \varphi} \text{)} \right)$, or $\frac{1}{na^r} \left(\cos. r\varphi . \right.$

$\left. l \,.\, (x^2 - 2ax \cos. \varphi + a^2) + 2 \sin. r\varphi \,.\, \text{arc} \right.$

$\left. (\text{tan.} \frac{x - a \cos. \varphi}{a \sin. \varphi} \text{)} \right)$.

If to the arc, whose tan. is $\frac{x - a \cos. \varphi}{a \sin. \varphi}$, the given arc, whose tan. is $\frac{\cos. \varphi}{\sin. \varphi}$, be added, we shall have an arc, whose tangent is $\frac{x \sin. \varphi}{a - \cos. \varphi}$, which may be used in place of the former, and which vanishes when $\varphi = o$.

A complex expression may sometimes be reduced, by proper substitution, to one of the integrable forms in the preceding examples: Thus,

EXAMPLE 16.

Given $y' = \frac{x^{n-1} \,.\, x'}{(1 - x^n) \,.\, (2x^n - 1)^{\frac{1}{2n}}}$. Put $v = (2x^n - 1)^{\frac{1}{2n}}$,

then $v^{2n} = 2x^n - 1$, and $v^{n-1} \cdot v' = x^{n-1} \cdot x'$; also $(1 - x^n) = \frac{1}{2}(1 - v^{2n})$. Hence $y' = \frac{v^{2n-1} \cdot v'}{\frac{1}{2}(1 - v^{2n})v}$ $= \frac{2v^{2n-1} \cdot v'}{1^{2n} - v^{2n}}$, a form similar to that in the preceding example.

EXAMPLE 17.

Given $y' = \frac{x'}{(1-x^n).(2x^n-1)^{\frac{1}{2n}}}$. Put $v = \frac{(2x^n-1)^{\frac{1}{2n}}}{x}$, then $v^{2n} = \frac{2x^n - 1}{x^{2n}}$, and $v^{2n-1} \cdot v' = -\frac{x'}{x^{n+1}} + \frac{x'}{x^{2n+1}}$ $= \frac{(1 - x^n)x'}{x^{2n+1}}$; also $1 - v^{2n} = 1 - \frac{2x^n - 1}{x^{2n}}$ $= \frac{(1 - x^n)^2}{x^{2n}}$. Now the preceding equation, divided by this, gives $\frac{v^{2n-1} \cdot v'}{1 - v^{2n}} = \frac{x'}{x(1 - x^n)}$; but $v = \frac{(2x^n-1)^{\frac{1}{2n}}}{x}$; therefore $\frac{v^{2n-2} \cdot v'}{1 - v^{2n}} = \frac{x'}{(1 - x^n).(2x^n-1)^{\frac{1}{2n}}}$.

IV. FORM, $\int \frac{x'}{\sqrt{A + Bx \pm Cx^2}}$.

EXAMPLE 18.

Given $y' = \frac{x'}{\sqrt{A + Bx + Cx^2}}$. Put the denomi-

nator $= c\sqrt{a+bx+x^2} = c(z-x)$; then $c = \sqrt{C}$, $a = \frac{A}{C}$, $b = \frac{B}{C}$, $x = \frac{z^2-a}{2z+b}$, and $x' = \frac{2zz'}{2z+b} - \frac{2z'(z^2-a)}{(2z+b)^2} = \frac{2z'(z^2+bz+a)}{(2z+b)^2}$; but $z - x = z - \frac{z^2-a}{2z+b} = \frac{z^2+bz+a}{2z+b}$; therefore $y' = \frac{x'}{c(z-x)} = \frac{2z'}{c(2z+b)}$, and $y = \frac{1}{c} l.(2z+b)$

$$= \frac{1}{c} l.\left(2\sqrt{a+bx+x^2} + 2x + b\right)$$

$$= \frac{1}{\sqrt{C}} l.\left(\frac{2}{\sqrt{C}}\sqrt{A+Bx+Cx^2} + \frac{B+2Cx}{C}\right).$$

Otherwise,

Let a, b, c be as before, but $x + \frac{1}{2}b = z$, and we have $x^2+bx = z^2-\frac{1}{4}b^2$. Hence $y' = \frac{z'}{c\sqrt{z^2+a-\frac{1}{4}b^2}}$,

and $y = \frac{1}{c} l.\left(z + \sqrt{z^2+a-\frac{1}{4}b^2}\right)$

$$= \frac{1}{c} l.\left(x+\tfrac{1}{2}b + \sqrt{x^2+bx+a}\right)$$

$$= \frac{1}{\sqrt{C}} l.\left(\frac{2Cx+B}{2C} + \frac{1}{\sqrt{C}}\sqrt{A+Bx+Cx^2}\right).$$

N. B.—This primitive exceeds the former by $\frac{1}{\sqrt{C}} l.2$, so that the variable part is the same in both.

EXAMPLE 19.

Given $y' = \frac{x'}{\sqrt{A + Bx - Cx^2}}$. Put a, b, c as before, and $x - \frac{1}{2}b = z$; then $x^2 - bx = z^2 - \frac{1}{4}b^2$, and $a + bx - x^2 = a + \frac{1}{4}b^2 - z^2 = d^2 - z^2$. Hence $y' = \frac{z'}{c\sqrt{d^2 - z^2}}$; consequently, $y = \frac{1}{c}$. arc $\left(\text{sin.} = \frac{z}{d}\right) = \frac{1}{\sqrt{C}}$. arc $\left(\text{sin.} = \frac{Cx - \frac{1}{2}B}{\sqrt{AC + \frac{1}{4}B^2}}\right)$.

In these two examples, if either A or B = o, we still have the primitives, for the particular cases, from the general expression; but if C = o, the whole expression vanishes, the primitive, in that case, being neither a logarithmic nor a circular, but an algebraic function. See example 10.

When the quantity under the radical sign, in these examples, can be resolved into two factors, the primitives may be easily found thus:

Suppose $\sqrt{a+bx+x^2} = \sqrt{m+x}\,.\,\overline{n+x} = (m+x)\,.\,z$; then $n + x = (m + x)\,.\,z^2$, and $x = \frac{mz^2 - n}{1 - z^2}$; also, $x' = \frac{2(m - n)zz'}{(1 - z^2)^2}$, and $\overline{m + x} = \frac{m - n}{1 - z^2}$.

Hence $\frac{x'}{\sqrt{a+bx+x^2}} = \frac{x'}{(m+x)\,z} = \frac{2z'}{1-z^2}$, and

$$y = l \cdot \frac{1+z}{1-z}.$$

Again, suppose $\sqrt{a+bx-x^2} = \sqrt{m-x \cdot n+x}$ $= \overline{m-x} \cdot z$; then, by a similar process, we find

$$\frac{x'}{\sqrt{a+bx-x^2}} = \frac{x'}{(m-x)\cdot z} = \frac{2z'}{1+z^2},$$

and $y = 2$ arc (tan. $= z$).

We shall now add some examples involving surd quantities, which cannot be reduced under any general form, and shew how to convert them into rational functions, whose primitives may be found.

EXAMPLE 20.

Given $y' = \frac{xx'}{\sqrt{1+x} - \sqrt[3]{1+x}}$. Here we may obtain a rational function, by substituting z^6 for $\overline{1+x}$; for then $y' = \frac{(z^6-1)6z^5z'}{z^3-z^2} = 6z^3z'\frac{z^6-1}{z-1} = 6z'$ $(z^8+z^7+z^6+z^5+z^4+z^3)$. Hence the primitive $y = 6\,(\frac{1}{9}z^9 + \frac{1}{8}z^8 + \frac{1}{7}z^7 + \frac{1}{6}z^6 + \frac{1}{5}z^5 + \frac{1}{4}z^4)$.

EXAMPLE 21.

Given $y' = \frac{x'}{\sqrt{1+x^2} - \sqrt{1-x^2}}$. To free the denominator of this fraction from surds, let both its terms be multiplied by $\sqrt{1+x^2} + \sqrt{1-x^2}$; then

$$y' = \frac{(\sqrt{1+x^2} + \sqrt{1-x^2}).x'}{2x^2}.$$

Put $z = \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{2x}$; then

$$z' = \frac{x'}{2\sqrt{1+x^2}} - \frac{x'}{2\sqrt{1-x^2}} - \frac{(\sqrt{1+x^2}+\sqrt{1-x^2})x'}{2x^2};$$

therefore $y' = \left(\frac{(\sqrt{1+x^2} + \sqrt{1-x^2})\,x'}{2x^2} = \right)$

$$\frac{x'}{2\sqrt{1+x^2}} - \frac{x'}{2\sqrt{1-x^2}} - z'; \text{ consequently,}$$

$$y = \tfrac{1}{2}\,l\,.\,(x + \sqrt{1+x^2}) - \tfrac{1}{2}\,\text{ar.}\,(\text{sin.} = x) - \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{2x}.$$

EXAMPLE 22.

Given $y' = (a + bx^n)^{\frac{p}{q}}\, x^m x'$; to find in what cases this may be converted into a rational function.

Put $(a + bx^n)^{\frac{p}{q}} = z^p$; then $a + bx^n = z^q$,

$$x^n = \frac{z^q - a}{b},\ x^m = \sqrt[n]{\frac{z^q - a}{b}}^m, \text{ and}$$

$$x^m x' = \frac{q}{nb} \cdot \overline{\frac{z^q - a}{b}}^{\frac{m+1}{n}-1} \cdot z^{q-1} \cdot z' ;$$

$$\text{therefore } y' = \frac{q}{nb} \cdot \overline{\frac{z^q - a}{b}}^{\frac{m+1}{n}-1} \cdot z^{p+q-1} \cdot z' ;$$

which is a rational function, when $\frac{m+1}{n}$ is a whole number.

If m, n be fractional numbers, as in this form, $(a + bx^{\frac{1}{2}})^{\frac{p}{q}} \cdot x^{-\frac{1}{2}} x'$, by putting $x = v^6$, we get $(a + bv^3)^{\frac{p}{q}} \cdot v^3 \cdot v'$, which is similar to the general form above.

Or, if n be negative, as in this form, $(a+bx^{-n})^{\frac{p}{q}} \cdot x^m x'$, suppose $x = \frac{1}{v}$; then $x' = -\frac{v'}{v^2}$, and $x^m = v^{-m}$, by which it may be changed to $-(a + bv^n)^{\frac{p}{q}} \cdot v^{-m-2} \cdot v'$.

Again, because $x^{\frac{np}{q}} (ax^{-n} + b)^{\frac{p}{q}} = (a + bx^n)^{\frac{p}{q}}$, we have also $y' = (b + ax^{-n})^{\frac{p}{q}} \cdot x^{m+\frac{np}{q}} \cdot x'$, which, by what has been already shewn, may be changed to a rational function, when $(m + \frac{np}{q} + 1)$, being divided by n, the quotient is a whole number.

Thus the given function may be converted into a rational one in these cases:

1. When $\frac{m+1}{n}$ = a whole number.

2. When $\frac{m+1}{n} + \frac{p}{q}$ = a whole number.

EXAMPLE 23.

To change $\sqrt{1-x^2}.x^2x'$ into a rational function.

Comparing this with the first of the preceding formulæ, we have $a = 1$, $b = -1$, $n = 2$, $p = 1$, $q = 2$, and $m = 2$. Here $\frac{m+1}{n} = \frac{3}{2}$ is not, but $\frac{m+1}{n} + \frac{p}{q} = 2$ is, a whole number. By the second formula, therefore, we have $y' = (b + ax^{-n})^{\frac{p}{q}}.x^{m+\frac{mp}{q}}.x'$ $= \left(-1 + \frac{1}{x^2}\right)^{\frac{1}{2}}.x^2.x'$, which, again, compared with the first, gives $a = -1$, $b = 1$, $p = 1$, $q = 2$, $n = -2$, and $m = 3$. Hence, putting $z^2 = \left(-1 + \frac{1}{x^2}\right)$, we get $y' = \frac{q}{nb}.\overline{\frac{z^2 - a}{b}}^{\frac{m+1}{n}-1}.z^{p+q-1}.z'$

$= -\frac{(z^2+1)^{-3}}{1}.z^2z' = -\frac{z^2z'}{(z^2+1)^3}$, a rational function.

In general, $\sqrt{1-x^2} \cdot x^m \cdot x'$ is convertible into a rational function, in all cases where m is a positive whole number; for, if m be odd, $\frac{m+1}{2}$ is a whole number; and, if m be even, $\frac{m+1}{2} + \frac{1}{2}$ is a whole number.

SECTION IX.

PARTIAL INTEGRATION.

THE following is a general method of investigating the primitive in certain forms of functions. The principle upon which it proceeds is this: The prime function of yv is $yv' + vy'$. Hence

$$\int yv' + \int vy' = yv,$$
$$\text{and } \int yv' = yv - \int vy'.$$

Thus we obtain a part of the integral, or primitive, of yv'; and, if $\int vy'$ can be resolved in a similar manner, we shall get another part, and so on, till the remainder shall have either a known primitive, or one that is too small to be regarded.

$\frac{m+1}{n} = r$, and $p + r = q$, viz.:

1. $\int z^p x^m x' = \frac{z^p x^{m+1}}{m+1} - \frac{bp}{r}\int z^{p-1} \cdot x^{m+n} \cdot x'.$

2. $\int z^p x^m x' = \frac{z^p x^{m+1}}{qn} + \frac{ap}{q}\int z^{p-1} \cdot x^m \cdot x'.$

3. $\int z^p x^m x' = \frac{z^{p+1} \cdot x^{m-n+1}}{bn(p+1)} - \frac{r-1}{b(p+1)}\int z^{p+1} \cdot x^{m-n} \cdot x'.$

4. $\int z^p x^m x' = -\frac{z^{p+1} \cdot x^{m+1}}{an(p+1)} + \frac{q+1}{a(p+1)}\int z^{p+1} \cdot x^m \cdot x'.$

5. $\int z^p x^m x' = \frac{z^{p+1} \cdot x^{m+1}}{a(m+1)} - \frac{b\overline{q+1}}{ar}\int z^p \cdot x^{m+n} \cdot x'.$

6. $\int z^p x^m x' = \frac{z^{p+1} \cdot x^{m-n+1}}{bnq} - \frac{a.\overline{r-1}}{bq}\int z^p \cdot x^{m-n} \cdot x'$

In applying these theorems to particular examples, it is easy to see, that, if a factor in the denominator of either term vanish, that term becomes infinite, and the theorem fails in that particular instance: Also, that, if a factor in the numerator of the second term vanish, the second term itself vanishes, and the first becomes the complete primitive, or, at least, the variable part of it.

EXAMPLE 1.

Required the primitive of $\frac{x^3 x'}{\sqrt{1-x^2}}$.

Here we have $(1-x^2)^{-\frac{1}{2}}.x^5x' = (a+bx^n)^p.x^m.x' = z^p.x^m.x'$.

Hence, $a=1$, $b=-1$, $n=2$, $p=-\frac{1}{2}$, $m=5$, $z=1-x^2$,

$$r = 3,\ q = 2\tfrac{1}{2}.$$

And as the index of x^5 can only be reduced by the third and sixth theorems, let us first apply the third.

$\int z^{-\frac{1}{2}}x^5x' = -\ z^{\frac{1}{2}}x^4 + 4\int z^{\frac{1}{2}}x^3x'$. In like manner,

$\int z^{\frac{1}{2}}x^3x' = -\frac{1}{3}z^{\frac{1}{2}}x^3 + \frac{3}{2}\int z^{\frac{1}{2}}xx'$. (where $m=3, r=2, p=\frac{1}{2}$).

$\int z^{\frac{1}{2}}xx' = -\frac{1}{3}z^{\frac{1}{2}}$. (where $m=1, r=1, p=\frac{3}{2}$).

Thus, the index 5 being reduced to 1, we get the integral of the last expression $=-\frac{1}{3}z^{\frac{1}{2}}$, because $r-1=o$ causes the other term to vanish. Hence the complete primitive is, $\int z^{-\frac{1}{2}}x^5x' = C - z^{\frac{1}{2}}x^4 - \frac{4}{5}z^{\frac{1}{2}}x^3 - \frac{8}{15}z^{\frac{1}{2}}$.

Again, by applying the sixth theorem, we find,

$$\int z^{-\frac{1}{2}}x^5x' = -\frac{1}{5}z^{\frac{1}{2}}x^4 + \frac{4}{5}\int z^{-\frac{1}{2}}x^3x',$$

$$\int z^{-\frac{1}{2}}x^3x' = -\frac{1}{3}z^{\frac{1}{2}}x^3 + \frac{3}{3}\int z^{-\frac{1}{2}}xx', \text{ and}$$

$$\int z^{-\frac{1}{2}}xx' = -z^{\frac{1}{2}}. \qquad \text{Consequently,}$$

$$\int z^{-\frac{1}{2}}x^5x' = C - \frac{1}{5}z^{\frac{1}{2}}x^4 - \frac{4}{3.5}z^{\frac{1}{2}}x^3 - \frac{2.4}{3.5}z^{\frac{1}{2}}.$$

And these two integrals are perfectly equal, as will easily appear by dividing each of them by $z^{\frac{1}{2}}$, and then substituting the values of z and x^2 in the former.

From the preceding example, it is evident that the integral of $\frac{x^m x'}{\sqrt{1-x^2}}$ may always be obtained by the third or sixth theorem, when m is an odd number; since, in the developement by either, the second term of the last partial integration vanishes.

When m is an even number, the integral of $\frac{x^m x'}{\sqrt{1-x^2}}$ may also be obtained, as the developement by it terminates in a known integral. Thus, the integral $\int z^{-\frac{1}{2}} x^4 x'$ will be found to terminate in $\int z^{-\frac{1}{2}} x' = \int \frac{x'}{\sqrt{1-x^2}}$, which is known to be the arc of a circle whose sine is x.

EXAMPLE 2.

To find the primitive of $\frac{x'}{x^4 . \sqrt{1-x^2}}$.

Here $(1-x^2)^{-\frac{1}{2}} . x^{-4} x' = (a+bx^n)^p . x^m . x' = z^p x^m x'$: Thus, $a=1$, $b=-1$, $n=2$, $p=-\frac{1}{2}$, $m=-4$, $r=-\frac{3}{2}$, and $q=-2$. And by theorem 5. the developement is as follows:

$$\int z^{-\frac{1}{2}} x^{-4} x' = -\tfrac{1}{3} z^{\frac{1}{2}} x^{-3} + \tfrac{2}{3} \int z^{-\frac{1}{2}} x^{-2} x',$$

$$\int z^{-\frac{1}{2}} x^{-2} x' = -z^{\frac{1}{2}} x^{-1} . \ (m=-2,\ r=-\tfrac{1}{2},\ q=-1),$$

the last term vanishing, because $q+1=0$.

Hence, $\int z^{-\frac{1}{2}} x^{-4} x' = C - \frac{1}{3} z^{\frac{1}{2}} x^{-3} - \frac{2}{3} z^{\frac{1}{2}} x^{-1}$.

EXAMPLE 3.

To find the primitive of $\frac{x'}{x^5\sqrt{1-x^2}}$.

By the same theorem, we have,

$$\int z^{-\frac{1}{2}}x^{-5}x' = -\tfrac{1}{4}z^{\frac{1}{2}}x^{-4} + \tfrac{3}{4}\int z^{-\frac{1}{2}}x^{-3}x', \text{ and}$$

$$\int z^{-\frac{1}{2}}x^{-3}x' = -\tfrac{1}{2}z^{\frac{1}{2}}x^{-2} + \tfrac{1}{2}\int z^{-\frac{1}{2}}x^{-1}x';$$

where the last term $\int z^{-\frac{1}{2}}x^{-1}x'$ or $\int \frac{x'}{x\sqrt{1-x^2}}$ is known to be the primitive of $-l.\frac{1+\sqrt{1-x^2}}{x}$. (D. 37. 18.)

Hence $\int z^{-\frac{1}{2}}x^{-5}x' = C - \frac{1}{4}z^{\frac{1}{2}}x^{-4} - \frac{1.3}{2.4}z^{\frac{1}{2}}x^{-2} - \frac{1.3}{2.4}l.\frac{1+\sqrt{1-x^2}}{x}$.

EXAMPLE 4.

To find the integral of $s^n x'$, s being the sine (and c the cosine) of the arc x, and n any positive whole number.

Put $s^{-n} = y$, then because $s' = cx'$, and $c' = -sx'$

$$y' = \overline{n-1}\,.\,s^{n-2}\,.\,cx',$$

and $\int s^n x' = \int -yc' = -yc + \int cy'$.

$$= -s^{n-1}.c + \overline{n-1}\int c^2 s^{n-2}x' =$$

(because $c^2 = 1-s^2$) $= -s^{n-1}.c + \overline{n-1}\int s^{n-2}x' - \overline{n-1}\int s^n x$.

Therefore $\int s^n x' = -\frac{1}{n} s^{n-1} c + \frac{n-1}{n} \int s^{n-2} . x'.$

Thus may the index n be reduced to zero, if it be an even number; or to 1, if it be odd: In the former case, the concluding term of the integral (without its co-efficient,) will be $\int x'$, or x; and in the latter, $\int s x'$, or c.

Hence this formula,

$$\int s^n x' = -\frac{1}{n} cs^{n-1} - \frac{n-1}{n.n-2} cs^{n-3} - \frac{n-1.n-3}{n.n-2.n-4} cs^{n-5} - \frac{n-1.n-3.n-5}{n.n-2.n-4.n-6} cs^{n-7} - \text{\&c.}$$

to (when n is even,) .... $+ \frac{n-1.n-3.n-5....1}{n.n-2.n-4....2} x + \text{C.}$

to (when n is odd,) .... $- \frac{n-1.n-3.n-5....2}{n.n-2.n-4.n-6.....1} c + \text{C.}$

EXAMPLE 5.

To find the integral of $c^n x'$. (c, s, x, n as before).

In a similar manner, we deduce the following formula:

$$\int c^n x' = \frac{1}{n} sc^{n-1} + \frac{n-1}{n.n-2} sc^{n-3} + \frac{n-1.n-3}{n.n-2.n-4} sc^{n-5} + \frac{n-1.n-3.n-5}{n.n-2.n-4.n-6} sc^{n-7} + \text{\&c.}$$

to (when n is even,) .... $+ \frac{n-1.n-3.n-5...1}{n.n-2.n-4...2} x + \text{C.}$

to (when n is odd,) .... $+ \frac{n-1.n-3.n-5.....2}{n.n-2.n-4.n-6...1} s + \text{C.}$

EXAMPLE 6.

Given $s^m c^n x'$, to find the primitive.

$$\int s^m c^n x' = \int s^{m-1} c^n . c' = -\frac{1}{n+1} c^{n+1} . s^{m-1} + \frac{m-1}{n+1} \int c^{n+2} . s^{m-2} . x'$$

$$= \text{(because } c^2 = 1 - s^2\text{)} = -\frac{1}{n+1} c^{n+1} . s^{m-1}$$

$$+ \frac{m-1}{n+1} \int c^n s^{m-2} . x' - \frac{m-1}{n+1} \int c^n s^m x'.$$

Therefore,

$$\int s^m c^n x' = -\frac{1}{m+n} c^{n+1} . s^{m-1} + \frac{m-1}{m+n} \int c^n s^{m-2} . x'. \quad \text{(A)}$$

In like manner,

$$\int s^m c^n x' = \frac{1}{m+n} s^{m+1} . c^{n-1} + \frac{n-1}{m+n} \int s^m c^{n-2} . x'. \quad \text{(B)}$$

By equation B,

$$\int s^{m-2} c^n x' = \frac{1}{m+n-2} s^{m-1} c^{n-1} + \frac{n-1}{m+n-2} \int s^{m-2} c^{n-2} x'.$$

By A and C,

$$\int s^m c^n x' = -\frac{1}{m+n} c^{n+1} s^{m-1} + \frac{m-1}{n+1 . m+n-2} s^{m-1} c^{n-1}$$

$$+ \frac{m-1 . n-1}{n+1 . m+n-2} \int s^{m-2} c^{n-2} x'.$$

Thus the indices m, n in the last term are both diminished by 2; and, by similar operations, they may be still farther depressed, till the whole integral shall be developed.

SECTION X.

INTEGRATION BY INFINITE SERIES.

We proceed to the general method of approximation by infinite series, and shall, in this section, consider a few of the numberless cases, in which a given prime function may be converted into an indefinite series of integrable terms; by actual division; by the binomial theorem; by the theorems of partial integration; or by those of polynomial derivation. And, since the logarithms of numbers, and the lengths of circular arcs, (which we have hitherto supposed to be given,) were originally computed by methods of approximation, and cannot indeed be computed otherwise, we shall begin with some examples of logarithmic and circular functions.

EXAMPLE 1.

Given $y' = \frac{x'}{a+x}$: Required y, or log. $(a+x)$.

Here $y' = \frac{x'}{a} - \frac{xx'}{a^2} + \frac{x^2x'}{a^3} - \frac{x^3x'}{a^4}$ + &c. (by division).

Therefore $y = \frac{x}{a} - \frac{1}{2}\cdot\frac{x^2}{a^2} + \frac{1}{3}\cdot\frac{x^3}{a^3} - \frac{1}{4}\cdot\frac{x^4}{a^4} + \&c.$

But, when $x = o$, $y\,(= l.(a+x))$ becomes $l.a$, which therefore is the constant quantity to be added to the above series, to complete the primitive of the given function.

Thus, $l.\overline{a+x} = l.a + \frac{x}{a} - \frac{1}{2}\cdot\frac{x^2}{a^2} + \frac{1}{3}\cdot\frac{x^3}{a^3} - \frac{1}{4}\cdot\frac{x^4}{a^4} + \frac{1}{5}\cdot\frac{x^5}{a^5} - \&c.$

And, $l.\overline{a-x} = l.a - \frac{x}{a} - \frac{1}{2}\cdot\frac{x^2}{a^2} - \frac{1}{3}\cdot\frac{x^3}{a^3} - \frac{1}{4}\cdot\frac{x^4}{a^4} - \frac{1}{5}\cdot\frac{x^5}{a^5} - \&c.$

Hence, $\frac{1}{2}\,l.\frac{a+x}{a-x} = \frac{x}{a} + \frac{1}{3}\cdot\frac{x^3}{a^3} + \frac{1}{5}\cdot\frac{x^5}{a^5} + \frac{1}{7}\cdot\frac{x^7}{a^7} + \&c.$

EXAMPLE 2.

Given $y' = \frac{ax'}{a^2+x^2}$: Required y, or the arc whose tangent is $\frac{x}{a}$ to the radius 1.

By division we obtain the two following series:

1. $\frac{ax'}{a^2+x^2} = \frac{x'}{a} - \frac{x^2x'}{a^3} + \frac{x^4x'}{a^5} - \frac{x^6x'}{a^7} + \frac{x^8x'}{a^9} - \&c.$

2. $\frac{ax'}{x^2+a^2} = \frac{ax'}{x^2} - \frac{a^2x'}{x^4} + \frac{a^3x'}{x^6} - \frac{a^7x'}{x^8} + \frac{a^9x'}{x^{10}} - \&c.$

To approximate the value of the primitive, we must take the *ascending* series, when x is less than a, that

is, when y is less than 45°; and the *descending* series, when x is greater than a, or y greater than 45°. In the former case we have,

$$y = \frac{x}{a} - \tfrac{1}{3}\cdot\frac{x^3}{a^3} + \tfrac{1}{5}\cdot\frac{x^5}{a^5} - \tfrac{1}{7}\cdot\frac{x^7}{a^7} + \tfrac{1}{9}\cdot\frac{x^9}{a^9} - \&c.$$

which requires no correction, as the primitive vanishes when $x = o$. But in the latter case,

$$y = -\frac{a}{x} + \tfrac{1}{3}\cdot\frac{a^3}{x^3} - \tfrac{1}{5}\cdot\frac{a^5}{x^5} + \tfrac{1}{7}\cdot\frac{a^7}{x^7} - \tfrac{1}{9}\cdot\frac{a^9}{x^9} + \&c.$$

where a correction is evidently required.

To find it, suppose $x = a$, then the two series should be equal: But, subtracting the latter, in that case, from the former, we find a difference equal to the double of the former; that is, = 90°, which therefore is the correction to be added to the latter.

Hence, $(90 - y) = \frac{a}{x} - \frac{1}{3}\cdot\frac{a^3}{x^3} + \frac{1}{5}\cdot\frac{a^5}{x^5} - \frac{1}{7}\cdot\frac{a^7}{x^7} + \&c.$

EXAMPLE 3.

Given $z' = \frac{s'}{\sqrt{a^2 - s^2}}$: Required z, or the arc whose sine is $\frac{s}{a}$ to the radius 1.

By the binomial theorem we find,

$$(a^2 - s^2)^{-\frac{1}{2}} = \frac{1}{a} + \tfrac{1}{2}\cdot\frac{s^2}{a^3} + \frac{1.3}{2.4}\cdot\frac{s^4}{a^5} + \frac{1.3.5}{2.4.6}\cdot\frac{s^6}{a^7} + \frac{1.3.5.7}{2.4.6.8}\cdot\frac{s^8}{a^9} + \&c.$$

which, multiplied by s', and integrated, gives,

$$z = \frac{s}{a} + \frac{1}{2.3}\cdot\frac{s^3}{a^3} + \frac{1.3}{2.4.5}\cdot\frac{s^5}{a^5} + \frac{1.3.5}{2.4.6.7}\cdot\frac{s^7}{a^7} + \&c.$$

where z and s vanish together.

When z is greater than 90°, but less than 180°, then $z' = \frac{(\sin. z)'}{-\cos. z} = -\frac{s'}{\sqrt{a^2-s^2}}$, which gives the same series with negative signs for the primitive, and therefore the addition of 90° (as the constant quantity,) is required.

When z is between 180° and 270°, we have $z' = \frac{-(\sin. z)'}{-(\cos. z)} = \frac{s'}{\sqrt{a^2-s^2}}$: The series is still the same, but the constant quantity 180°, &c.

EXAMPLE 4.

Given $y' = \frac{x'}{\sqrt{1+x^2}}$, to find y or $l.\,(x + \sqrt{1+x^2})$.

By the same theorem we have,

$$y' = x' - \tfrac{1}{2}.x^2x' + \frac{1.3}{2.4}x^4x' - \frac{1.3.5}{2.4.6}.x^6x' + \&c.$$

therefore,

$$y = x - \frac{1}{2.3}x^3 + \frac{1.3}{2.4.5}x^5 - \frac{1.3.5}{2.4.6.7}x^7 + \frac{1.3.5.7}{2.4.6.8.9}x^9 - \&c.$$

Hence, if $x = \frac{m-n^2}{2mn}$, then $(x+\sqrt{1+x^2}) = \frac{m}{n}$; so

that the above series gives the log. $\frac{m}{n}$, when $x = \frac{m^2 - n^2}{2mn}$.

EXAMPLE 5.

Given $y' = \frac{x'}{\sqrt{x - x^2}}$: Required y.

By putting $v^2 = x$, we get $y' = \frac{2vv'}{v\sqrt{1-v^2}} = \frac{2v'}{\sqrt{1-v^2}}$, therefore, according to the third example, we have,

$$y = 2\left(v + \frac{1}{2.3}v^3 + \frac{1.3}{2.4.5}v^5 + \frac{1.3.5}{2.4.6.7}v^7 + \text{\&c.}\right)$$
$$= 2\sqrt{x}\left(1 + \frac{1}{2.3}x + \frac{1.3}{2.4.5}x^2 + \frac{1.3.5}{2.4.6.7}x^3 + \text{\&c.}\right)$$

EXAMPLE 6.

Given $y' = \frac{x'}{\sqrt{x^2 - 1}}$: Required y, or the $l.(x + \sqrt{x^2 - 1})$.

By putting $x = \sqrt{1 + 2v}$, we get $y' = \frac{v'}{\sqrt{v + v^2}}$, which, expanded by the binomial theorem, and integrated, gives

$$y = 2\sqrt{v}\left(1 - \frac{1}{2.3}v + \frac{1.3}{2.4.5}v^2 - \frac{1.3.5}{2.4.6.7}v^3 + \text{\&c.}\right)$$

EXAMPLE 7.

Given $y' = (a + bx^n)^p x^m x'$, to find y.

Expanding $(a + bx^n)^p$ by the binomial theorem, multiplying by $x^m x'$, and integrating, we find, (putting $m + n + 1 = t$,)

$$y = a^p \left(\frac{x^{m+1}}{m+1} + \frac{p}{t} \cdot \frac{b}{a} x^t + \frac{p \cdot p-1}{1 \cdot 2 (t+n)} \cdot \frac{b^2}{a^2} x^{t+n} + \frac{p \cdot p-1 \cdot p-2}{1 \cdot 2 \cdot 3 (t+2n)} \cdot \frac{b^3}{a^3} x^{t+2n} + \text{&c.} \right)$$

Also, because $(a + bx^n)^p x^m x' = b^p \left(1 + \frac{ax^{-n}}{b}\right)^p . x^{m+np} . x$,

we find, in the same manner, (putting $m + pn + 1 = q$,)

$$y = b^p \left(\frac{x^q}{q} - \frac{p}{q-n} \cdot \frac{a}{b} x^{q-n} + \frac{p \cdot p-1}{1.2.(q-2n)} \cdot \frac{a^2}{b^2} x^{q-2n} - \frac{p \cdot p-1 \cdot p-2}{1 \cdot 2 \cdot 3 \cdot (q-3n)} \cdot \frac{a^3}{b^3} x^{q-3n} + \text{&c.} \right)$$

This general expression of a derivative function, viz.

$$(a + bx^n)^p . x^m . x' \text{ (or } z^p x^m x'),$$

may be thrown into an indefinite series of integrals, by means of the theorems of partial integration. Thus, by theorem 1,

$$\int z^p x^m x' = \frac{z^p x^{m+1}}{m+1} - \frac{bnp}{m+1} \int z^{p-1} . x^{m+n} . x',$$

in which, substituting $p - 1$ for p, and $m + n$ for m, it becomes,

$$\int z^{p-1} . x^{m+n} . x' = \frac{z^{p-1} x^{m+n+1}}{m+n+1} - \frac{bn . \overline{p-1}}{m+n+1} \int z^{p-2} . x^{m+2n} . x'.$$

$$\text{Hence } \frac{bnp}{m+1}\int z^{p-1}\,.\,x^{m+n}\,.\,x' = \frac{bpz^{p-1}\,.\,x^{m+n+1}}{m+1\,.\,r+1}$$

$$-\frac{b^2np\,.\,p-1}{m+1\,.\,r+1}\int z^{p-2}\,.\,x^{m+2n}\,.\,x',$$

$$\text{and } \int z^p x^m x' = \frac{z^p x^{m+1}}{m+1} - \frac{bp\,.\,z^{p-1}\,.\,x^{m+n+1}}{m+1\,.\,r+1}$$

$$+\frac{b^2np\,.\,p-1}{m+1\,.\,r+1}\int z^{p-2}\,.\,x^{m+2n}\,.\,x'.$$

Again,

$$\int z^{p-2}\,.\,x^{m+2n}\,.\,x' = \frac{z^{p-2}\,.\,x^{m+2n+1}}{m+2n+1} - \frac{bn.p-2}{m+2n+1}\int z^{p-3}\,.\,x^{m+3n}x'.$$

Hence

$$\frac{b^2np\,.\,p-1}{m+1\,.\,r+1}\int z^{p-2}\,.\,x^{m+2n}\,.\,x' = \frac{b^2p.\overline{p-1}.z^{p-2}.x^{m+2n+1}}{m+1\,.\,r+1\,.\,r+2}$$

$$-\frac{b^3np.p-1.p-2}{m+1\,.\,r+1\,.\,r+2}\int z^{p-3}\,.\,x^{m+3n}\,.\,x',$$

$$\text{and } \int z^p x^m x' = \frac{z^p x^{m+1}}{m+1} - \frac{bpz^{p-1}.x^{m+n+1}}{m+1\,.\,r+1}$$

$$+\frac{b^2p.p-1z^{p-2}.x^{m+2n+1}}{m+1\,.\,r+1\,.\,r+2} - \frac{b^3np.p-1.p-2}{m+1.r+1.r+2}\int z^{p-3}.x^{m+3n}x'.$$

And so on: Proceeding in this manner with each of the said theorems, we obtain the following series, each of which expresses the integral of $(a+bx^n)^p\,.\,x^m.\,x'$.

Putting $z = (a+bx^n)$, $r = \dfrac{m+1}{n}$, and $q = \overline{r+p.}$

1. $$\frac{z^p x^{m+1}}{m+1}\left(1-\frac{p}{r+1}\cdot\frac{bx^n}{z}+\frac{p.p-1}{r+1.r+2}\cdot\frac{b^2x^{2n}}{z^2}\right.$$
$$\left.-\frac{p.p-1.p-2}{r+1.r+2.r+3}\cdot\frac{b^3x^{3n}}{z^3}+\&c.\right)$$

2. $$\frac{z^p x^{m+1}}{qn}\left(1+\frac{p}{q-1}\cdot\frac{a}{z}+\frac{p.p-1}{q-1.q-2}\cdot\frac{a^2}{z^2}\right.$$
$$\left.+\frac{p.p-1.p-2}{q-1.q-2.q-3}\cdot\frac{a^3}{z^3}+\&c.\right)$$

3. $$\frac{z^{p+1}x^{m-n+1}}{bn.p+1}\left(1-\frac{r-1}{p+2}\cdot\frac{z}{bx^n}+\frac{r-1.r-2}{p+2.p+3}\cdot\frac{z^2}{b^2x^{2n}}\right.$$
$$\left.-\frac{r-1.r-2.r-3}{p+2.p+3.p+4}\cdot\frac{z^3}{b^3x^{3n}}+\&c.\right)$$

4. $$-\frac{z^{p+1}x^{m+1}}{an.p+1}\left(1+\frac{q+1}{p+2}\cdot\frac{z}{a}+\frac{q+1.q+2}{p+2.p+3}\cdot\frac{z^2}{a^2}\right.$$
$$\left.+\frac{q+1.q+2.q+3}{p+2.p+3.p+4}\cdot\frac{z^3}{a^3}+\&c.\right)$$

5. $$\frac{z^{p+1}.x^{m+1}}{a.m+1}\left(1-\frac{q+1}{r+1}\cdot\frac{bx^n}{a}+\frac{q+1.q+2}{r+1.r+2}\cdot\frac{b^2x^{2n}}{a^2}\right.$$
$$\left.-\frac{q+1.q+2.q+3}{r+1.r+2.r+3}\cdot\frac{b^3x^{3n}}{a^3}+\&c.\right)$$

6. $$\frac{z^{p+1}.x^{m-n+1}}{bmq}\left(1-\frac{r-1}{q-1}\cdot\frac{a}{bx^n}+\frac{r-1.r-2}{q-1.q-2}\cdot\frac{a^2}{b^2x^{2n}}\right.$$
$$\left.-\frac{r-1.r-2.r-3}{q-1.q-2.q-3}\cdot\frac{a^3}{b^3x^{3n}}+\&c.\right)$$

To these series we may add another deduced from the first, by multiplying it by $\frac{z-bx^n}{a}(=1)$, viz.

7. $$\frac{z^{p+1}x^{m+1}}{a.m+1}\left(1-\frac{q+1}{r+1}\cdot\frac{bx^n}{az}+\frac{q+1.p}{r+1.r+2}\cdot\frac{b^2x^{2n}}{a^2z^2}\right.$$
$$\left.-\frac{q+1.p.p-1}{r+1.r+2.r+3}\cdot\frac{b^3x^{3n}}{a^3z^3}+\&c.\right)$$

In the application of these series, it is to be observed,

1. That if a factor in the numerator of the second, or any subsequent term vanish, the exact primitive, (at least the variable part of it,) will be obtained.

2. That if a factor in the denominator of any term vanish, the method by that series totally fails.

Thus, if p be any positive whole number, the first and second series will terminate; if r be any positive whole number, (that is, if $m+1$ be equal to n, or to any multiple of n,) the third and sixth will terminate; or, if q be any negative whole number, the fourth and fifth will terminate: But if p be a negative whole number, the third and fourth series fail; if r be a negative whole number, the first and fifth fail; or, if q be a positive whole number, (or $= o$,) the second and sixth fail.

3. In the case in which $z = o$, each series fails, (except the last two, when $p = -1$); yet the integral may also be found for that case, as shall be shewn in the next Section.

4. Any two of these series are either equal to each other, or differ from each other by some constant quantity.

EXAMPLE 1.

Given $\frac{x'}{1+x}$, to approximate the primitive $(=l.\overline{1+x})$.

Here $(a+bx^n)^p . x^m . x' = (1+x)^{-1} . x^0 . x'$. Hence $z = \overline{1+x}$, $a = 1$, $b = 1$, $n = 1$, $p = -1$, $m = o$, $r = 1$, and $q = o$.

The third and fourth series fail, because $p = -1$; also, the second and sixth, because $q = o$. From the first and fifth we have,

1. $\int \frac{x'}{1+x} = \frac{x}{z} + \frac{1}{2}.\frac{x^2}{z^2} + \frac{1}{3}.\frac{x^3}{z^3} + \frac{1}{4}.\frac{x^4}{z^4} + \frac{1}{5}.\frac{x^5}{z^5} + \&c.$

5. $\int \frac{x'}{1+x} = x - \frac{1}{2}.x^2 + \frac{1}{3}.x^3 - \frac{1}{4}.x^4 + \frac{1}{5}.x^5 - \&c.;$

of which the latter is the common series for $l.\overline{1+x}$, and may be obtained in various ways. But the former is so much more convergent, that, in finding the log. 2, by making $x = 1$, 20 terms only would be equivalent to 100,000 of the latter.

The seventh series is still more convergent. It gives,

7. $\int \frac{x'}{1+x} = x - \frac{1}{1.2}.\frac{x^2}{z} - \frac{1}{2.3}.\frac{x^5}{z^2} - \frac{1}{3.4}.\frac{x^7}{z^3} - \frac{1}{4.5}.\frac{x^9}{z^4} - \&c.$

EXAMPLE 2.

To approximate the value of $\int \frac{x'}{1+x^2}$, or an arc of a circle whose tangent is x. to the radius 1.

$$(a+bx^n)^p \cdot x^m \cdot x = (1+x^2)^{-1} \cdot x^0 \cdot x'.$$

$a = 1$, $b = 1$, $n = 2$, $p = -1$, $m = o$, $r = \frac{1}{2}$, $q = -\frac{1}{2}$, and $z = \sqrt{1+x^2}$.

Because $p = -1$, the third and fourth series fail; but, taking the others in their order, we find,

1. $$\int \frac{x'}{1+x^2} = \frac{x}{z} + \frac{2}{3} \cdot \frac{x^3}{z^3} + \frac{2.4}{3.5} \cdot \frac{x^5}{z^5} + \frac{2.4.6}{3.5.7} \cdot \frac{x^7}{z^7} + \frac{2.4.6.8}{3.5.7.9} \cdot \frac{x^9}{z^9} + \text{&c.}$$

2. $$\int \frac{x'}{1+x^2} = -\frac{x}{z} - \frac{2}{3} \cdot \frac{x}{z^3} - \frac{2.4}{3.5} \cdot \frac{x}{z^5} - \frac{2.4.6}{3.5.7} \cdot \frac{x}{z^7} - \text{&c.}$$

3. $$\int \frac{x'}{1+x^2} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \text{&c.}$$

6. $$\int \frac{x'}{1+x^2} = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \frac{1}{9x^9} + \frac{1}{11x^{11}} - \text{&c.}$$

The second and last of these integrals evidently require a correction. Let Q be the constant quantity, which, added to the last, completes the primitive (y); then

$$y = Q - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \text{&c.}; \text{ and}$$

$$Q - y = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \text{&c.}$$

But $\frac{1}{x} = \text{cot. } y$. Hence it appears, from the preceding series, that $Q - y$ must be the complement of y; therefore $Q = 90°$. Thus, the signs of the second and sixth series being all changed, each series becomes the correct primitive of the complemental arc. There are, in fact, therefore, only two methods of computation here for the arc of a circle, as there were in the preceding example for the logarithm of a number; and here, too, when $x = 1$, the first series is by far the most convergent of the two, though the seventh series is still more so.

$$7.\ \int \frac{x'}{1+x^2} = x - \tfrac{1}{3} \cdot \frac{x^3}{z} - \frac{2}{3.5} \cdot \frac{x^5}{z^3} - \frac{2.4}{3.5.7} \cdot \frac{x^7}{z^5} - \frac{2.4.6}{3.5.7.9} \cdot \frac{x^9}{z^4} - \&c.$$

Or, putting $x = \frac{1}{v}$,

$$\int \frac{x'}{1+x^2} = \frac{1}{v} - \frac{1}{3v(v^2+1)} - \frac{2}{3.5v(v^2+1)^2} - \frac{2.4}{3.5.7v(v^2+1)^3} - \&c.$$

Thus, if tan. A $= \frac{1}{7}$, and tan. B $= \frac{2}{11}$, then

$$45° = \overline{3A + 2B}.$$

$$45^\circ = \frac{3}{7} - \left(\frac{2}{700} + \frac{4a}{500} + \frac{8b}{700} + \frac{12c}{900} + \frac{16d}{1100} + \&c.\right)$$
$$+ \frac{4}{11} - \left(\frac{128}{33000} + \frac{2.32a}{5000} + \frac{4.32b}{7000} + \frac{6.32c}{9000} + \&c.\right)$$

where a, b, c, &c. denote each the preceding term; that is, $a = \frac{2}{700}$, $b = \frac{4a}{500}$, &c. in the first series; and $a = \frac{128}{33000}$, $b = \frac{2.32}{5000}$, &c. in the second.

EXAMPLE 3.

To approximate the value of $\int \frac{x'}{1-x^2} \left(= \frac{1}{2}l.\frac{1+x}{1-x}\right)$.

By applying the same theorems, as in the preceding example, we obtain the very same series, the signs of the even terms being changed.

1. $\int \frac{x'}{1-x^2} = \frac{x}{z} - \frac{2}{3}\cdot\frac{x^3}{z^3} + \frac{2.4}{3.5}\cdot\frac{x^5}{z^5} - \frac{2.4.6}{3.5.7}\cdot\frac{x^7}{z^7} + \&c.$

5. $\int \frac{x'}{1-x^2} = x + \frac{1}{3}.x^3 + \frac{1}{5}.x^5 + \frac{1}{7}.x^7 + \frac{1}{9}.x^9 + \&c.$

7. $\int \frac{x'}{1-x^2} = x + \frac{1}{3}.\frac{x^3}{z} - \frac{2}{3.5}\cdot\frac{x^5}{z^5} + \frac{2.4}{3.5.7}\cdot\frac{x^7}{z^7} - \&c.$

The following is an example of an infinite series of integrals obtained by polynomial derivation.

EXAMPLE.

To find the integral of $\frac{(1-c^2x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}} \cdot x\,(=y')$.

Let the fraction be converted into the series,

$$A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + \&c. \quad (D.75.\ ex.\ 2.)$$

Thus, $A = 1$.

$B = \frac{1}{2}(1-c^2)$.

$C = \frac{1}{8}(3-2c^2-c^4)$.

$D = \frac{1}{32}(10-6c^2-2c^4-2c^6)$.

$E = \frac{1}{128}(35-20c^2-6c^4-4c^6-5c^8)$.

&c. &c.

$$\text{Hence } y = x + \frac{1-c^2}{6}x^3 + \frac{3-2c^2-c^4}{40}x^5 + \frac{10-3c^2-c^4-c^6}{224}x^7 + \&c.$$

This, when $c = o$, is the series for (z), the arc of a circle, of which x is the sine and 1 the radius; therefore,

$$z-y = \frac{c^2}{2.3}x^3 + \frac{2c^2+c^4}{2^3.5}x^5 + \frac{6c^2+2c^4+2c^6}{2^5.7}x^7 + \frac{20c^2+6c^4+4c^6+5c^8}{2^7.9}x^9 + \&c.$$

The law of continuation will be manifest from the following table:

$$
\begin{array}{l}
c^2. \\
2c^2 + \quad c^4. \\
6c^2 + \ 2c^4 + \ 2c^6. \\
20c^2 + \ 6c^4 + \ 4c^6 + \ 5c^8. \\
70c^2 + 20c^4 + 12c^6 + 10c^8 + 14c^{10}. \\
252c^2 + 70c^4 + 40c^6 + 30c^8 + 28c^{10} + 42c^{12} \\
\&c. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \&c.
\end{array}
$$

The coefficients in the first perpendicular column are the numbers of the middle diagonal line in the table of figurate numbers. Those in the other perpendicular columns are multiples of the former by 1, 2, 5, 14, 42, &c. which are the differences between the doubles of the numbers in the said diagonal line, and the corresponding numbers in the diagonal line adjacent to it.

Hence, if

$$
\begin{array}{lll}
A = & \frac{1}{2.3} + \frac{2}{2^3.5} + \frac{6}{2^5.7} + \frac{20}{2^7.9} + \frac{70}{2^9.11} + \frac{252}{2^{11}.13} + \&c. \\
B = & \qquad \frac{1}{2^3.5} + \frac{2}{2^5.7} + \frac{6}{2^7.9} + \frac{20}{2^9.11} + \frac{70}{2^{11}.13} + \&c. \\
C = & \qquad\qquad \frac{2}{2^5.7} + \frac{4}{2^7.9} + \frac{12}{2^9.11} + \frac{40}{2^{11}.13} + \&c. \\
D = & \qquad\qquad\qquad \frac{5}{2^7.9} + \frac{10}{2^9.11} + \frac{30}{2^{11}.13} + \&c. \\
\&c. & \qquad\qquad\qquad\qquad \&c.
\end{array}
$$

Then, (when $x = 1$, and $z = Q$,)

$$\overline{Q - y} = Ac^2 + Bc^4 + Cc^6 + Dc^8 + Ec^{10} + \&c.$$

Otherwise,

Putting $1-c^2=b^2$, we have $\left(\frac{1-c^2x^2}{1-x^2}\right)^{\frac{1}{2}} = \left(c^2+\frac{b^2}{1-x^2}\right)^{\frac{1}{2}}$

$$= (c^2 + b^2 + b^2x^2 + b^2x^4 + b^2x^6 + \&c.)^{\frac{1}{2}}$$

$$= (1 + b^2x^2 + b^2x^4 + b^2x^6 + \&c.)^{\frac{1}{2}}$$

$= A + Bx^2 + Cx^4 + Dx^6 + \&c.$ (See D. 57. ex. 4.)

$A = 1.$

$B = \frac{1}{2}b^2.$

$C = \frac{1}{8}(4b^2 - b^4).$

$D = \frac{1}{32}(16b^2 - 8b^4 + 2b^6).$

&c. &c.

Thus, $y = x + \frac{1}{2.3}b^2x^3 + \frac{1}{2^3.5}(4b^2 - b^4)x^5$

$$+ \frac{1}{2^5.7}(16b^2 - 8b^4 + 2b^6)x^7 + \&c.$$

which, since $b^2 = 1 - c^2$, is the same as before.

SECTION XI.

INTEGRATION OF $(a - bx^n)^p x^m x'$, WHEN $a - bx^n = 0$.

By reversing the fifth theorem, (where $\frac{m+1}{n} = r$, and $r + p + 1 = t$,) we get,

$$\int z^p x^{m+n} x' = \frac{z^{p+1} \cdot x^{rn}}{bnt} - \frac{ar}{bt} \int z^p x^m x' ;$$

in which, substituting $m + n$ for m, and, consequently, $r + 1$ for r, and $t + 1$ for t,

$$\int z^p x^{m+2n} x' = \frac{z^{p+1} \cdot x^{rn+n}}{bn \cdot t + 1} - \frac{a \cdot r + 1}{b \cdot t + 1} \int z^p x^{m+n} x',$$

$$\int z^p x^{m+3n} x' = \frac{z^{p+1} \cdot x^{rn+2n}}{bn \cdot t + 2} - \frac{a \cdot r + 2}{b \cdot t + 2} \int z^p x^{m+2n} x',$$

&c. &c.

Hence,

$$\int z^p x^{m+vn} x' = \frac{z^{p+1} \cdot x^{rn+vn-n}}{bn \cdot t+v-1} - \frac{\overline{a \cdot r+v-1}}{\overline{b \cdot t+v-1}} \cdot \frac{z^{p+1} x^{rn+vn-2n}}{bn \cdot t+v-2}$$

$$+ \frac{a^2 \cdot \overline{r+v-1} \cdot \overline{r+v-2}}{b^2 \cdot \overline{t+v-1} \cdot \overline{t+v-2}} \cdot \frac{z^{p+1} \cdot x^{rn+vn-3n}}{bn \cdot t+v-3} - \&c.$$

$$\pm \frac{a^v}{b^v} \cdot \frac{r \cdot r+1 \cdot r+2 \ldots \overline{r+v-1}}{t \cdot t+1 \cdot t+2 \ldots \overline{t+v-1}} \int z^p x^m x' ;$$

where v may be any positive whole number; and the last term is affirmative or negative, according as v is even or odd: But when b is negative, the last term is affirmative, whether b be even or odd; and if, at the same time, $z = o$, it is the only term that remains on one side of the equation, (except when $p = -1$). In that case, therefore,

$$\int z^p x^{m+nv} x' = \frac{a^v}{b^v} \cdot \frac{r \cdot r+1 \cdot r+2 \ldots r+v-1}{t \cdot t+1 \cdot t+2 \ldots t+v-1} \int z^p x^m x'.$$

And, in the same case,

$$\int z^p x^m x' \left(\alpha + \epsilon x^n + \gamma x^{2n} + \delta x^{3n} + \&c.\right)$$
$$= \int z^p x^m x' \cdot \left(\alpha + \frac{a}{b} \cdot \frac{r}{t} \epsilon + \frac{a^2}{b^2} \cdot \frac{r \cdot r+1}{t \cdot t+1} \gamma \right.$$
$$\left. + \frac{a^3}{b^3} \cdot \frac{r \cdot r+1 \cdot r+2}{t \cdot t+1 \cdot t+2} \cdot \delta + \&c.\right)$$

where $z = a - bx^n = o$, $r = \frac{m+1}{n}$, and $t = r+p+1$.

EXAMPLE 1.

Required the integrals of $\frac{x^2 x'}{\sqrt{1-x^2}}$, $\frac{x^4 x'}{\sqrt{1-x^2}}$, &c. when $x = 1$.

Here $p = -\frac{1}{2}$, $m = o$, $n = 2$, $r = \frac{1}{2}$, and $t = 1$.

Therefore, when $v = 1$, $\int \frac{x^2 x'}{\sqrt{1-x^2}} = \frac{1}{2} \int \frac{x'}{\sqrt{1-x^2}} = \frac{1}{2} Q$;

when $v=2, \int \frac{x^4x'}{\sqrt{1-x^2}} = \frac{3}{8}\,Q$;

when $v=3, \int \frac{x^6x'}{\sqrt{1-x^2}} = \frac{5}{16}\,Q$,

&c. &c.

In general, $\int \frac{x^{2v}x'}{\sqrt{1-x^2}} = \frac{1.3.5.7 \ldots\ldots \overline{2v-1}}{2.4.6.8 \ldots\ldots 2v}\,Q.$

EXAMPLE 2.

To find the integral of $\frac{(1-c^2x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}} \cdot x'$, when $1-x^2=o$.

We have $\frac{(1-c^2x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}\,x' = \frac{x'}{\sqrt{1-x^2}} \cdot \sqrt{1-c^2x^2}$

$$= \frac{x'}{\sqrt{1-x^2}}\left(1-\tfrac{1}{2}c^2x^2-\frac{1}{2.4}c^4x^4 - \frac{1.3}{2.4.6}c^6x^6 - \&c.\right.$$

Therefore,

$$\int \left(\frac{1-c^2x^2}{1-x^2}\right)^{\frac{1}{2}} \cdot x' = Q\left(1 - \tfrac{1}{2}\cdot\tfrac{1}{2}c^2 - \frac{1.3}{2.4}\cdot\frac{1}{2.4}\,c^4 - \frac{1.3.5}{2.4.6}\cdot\frac{1.3}{2.4.6}\,c^6x^6 - \frac{1.3.5.7}{2.4.6.8}\cdot\frac{1.3.5}{2.4.6.8}\,c^8x^8 - \frac{1..9}{2..10}\cdot\frac{1..7}{2..10}\,c^{10}x^{10} - \&c.\right)$$

From the preceding example, it will easily appear that this integral is the very same with that which was obtained at the end of the tenth section, on the supposition of $x = 1$.

Again, supposing $m+1=n$, we have by theorem 3.

$$\int z^n x^{n-1} x' = -\frac{z^{p+1}}{bn.p+1}$$ (b being negative, and $z = o$.)

To complete the primitive, a constant quantity (found by putting $x=o$,) must be added; then,

$$\int z^p x^{n-1} x' = \left(-\frac{z^{p+1}}{bn.p+1} + \frac{a^{p+1}}{bn.p+1}\right) = \text{P suppose.}$$

Hence,

$$\int z^p x^{2n-1} x' = \left(-\frac{z^{p+1}x^n}{bn.p+2} + \frac{a}{b}\cdot\frac{1}{p+2}\,\text{P}\right) = \text{Q.}\quad \text{By eq. 1.}$$

$$\int z^p x^{3n-1} x' = \left(-\frac{z^{p+1}x^{2n}}{bn.p+3} + \frac{a}{b}\cdot\frac{2}{p+3}\,\text{Q}\right) = \text{R.}$$

$$\int z^p x^{4n-1} x' = \left(-\frac{z^{p+1}x^{3n}}{bn.p+4} + \frac{a}{b}\cdot\frac{3}{p+4}\,\text{R}\right) = \text{\&c.}$$

Therefore, putting s for any positive whole number,

$$\int z^p x^{sn-1} x' = -\frac{z^{p+1}x^{sn-n}}{bn.p+s} - \frac{a}{b^2n}\cdot\frac{\overline{s-1}.z^{p+1}.x^{sn-2n}}{p+s.p+s-1}$$

$$-\frac{a^2}{b^3n}\cdot\frac{\overline{s-1}.\overline{s-2}z^{p+1}x^{sn-3n}}{p+s.p+s-1.p+s-2} - \text{\&c. .....}$$

$$+\frac{a^{p+s}}{nb^s}\left(\frac{1.2.3.....\overline{s-1}}{p+1.p+2.p+3...p+s}\right).$$

Consequently, when $z = o$,

$$\int z^p x^{sn-1} x' = \frac{a^{p+s}}{snb^s}\left(\frac{1.2.3.....s}{p+1.p+2.p+3...p+s}\right) = \text{C.}$$

Now, if s be increased by 0, 1, 2, 3, &c. successively, that is, if $z^p x^{sn-1} x'$ be multiplied by the several

terms of the series 1, x^n, x^{2n}, x^{3n}, &c., we shall have for the corresponding primitives, (putting $\overline{s+p+1}=r$),

$$C;\ \frac{a}{b}\cdot\frac{s}{r}C;\ \frac{a^2}{b^2}\cdot\frac{s.s+1}{r.r+1}C;\ \frac{a^3}{b^3}\cdot\frac{s.s+1.s+2}{r.r+1.r+2}C;\ \&c.$$

But in the series 1, $\frac{s}{r}$, $\frac{s.s+1}{r.r+1}$, $\frac{s.s+1.s+2}{r.r+1.r+2}$, &c., if d denote the distance of any term from 1, we have the following general expression for *that* term, whether d be integral or fractional, affirmative or negative, viz. ($r-s$ being equal to $p+1$.)

$$1-\frac{d}{1}\cdot\frac{p+1}{r}+\frac{d.d-1}{1.2}\cdot\frac{p+1.p+2}{r.r+1}$$
$$-\frac{d.d-1.d-2}{1.2.3}\cdot\frac{p+1.p+2.p+3}{r.r+1.r+2}+\&c.$$

therefore,

$$\int x^p x^{sn+dn-1}x' = \frac{a^d}{b^d}C\left(1-\frac{d}{1}\cdot\frac{p+1}{r}+\frac{d.d-1}{1.2}\cdot\frac{p+1.p+2}{r.r+1}-\&c.\right)$$

Hence, putting $m=sn+dn-1$, or $\frac{m+1}{n}=s+d$, and assuming $s=$ any whole number, the value of d will be had.

EXAMPLE.

To find $\int\frac{x'}{\sqrt{1-x^2}}$, when $x=1$; or the length of a quadrant, of which 1 is the radius,

Here, $x^p x^{sn+dn-1}x'=(1-x^2)^{-\frac{1}{2}}.x^0.x'$; $p=-\frac{1}{2}$, $n=2$,

$m = o$, $s + d = \frac{1}{2}$, suppose $s = 1$, then $d = -\frac{1}{2}$, $r = s+p+1 = \frac{3}{2}$, $a = b = 1$, and $C = \frac{1}{n.p+1}$. Therefore,

$$\int \frac{x^r}{\sqrt{1-x^2}} = 1 - \frac{d}{1} \cdot \frac{p+1}{r} + \frac{d.d-1}{1.2} \cdot \frac{p+1.p+2}{r.r+1}$$
$$- \frac{d.d-1.d-2}{1.2.3} \cdot \frac{p+1.p+2.p+3}{r.r+1.r+2} + \&c.$$
$$= 1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.6.7} + \frac{1.3.5.7}{2.4.6.8.9} + \&c.$$

which is the well-known series for 90°, of which 1 is the sine: But not sufficiently convergent for use. The preceding method, however, may be improved as follows:

Since $\frac{m+1}{n} = r$, $m = rn-1$; therefore, when $z = o$,

$$\int z^p x^{m+rn-1} x^r = \frac{a^v}{b^v} \cdot \frac{r.r+1.r+2 \ldots\ldots r+v-1}{t.t+1.t+2 \ldots\ldots t+v-1} \int z^p x^m x^r.$$

Now, if we put $v + r = s + d$, it is evident that this integral will be equivalent to the former. And thence we deduce the following,

$$\int z^p x^m x^r = \frac{a^r}{snb^r} \left(\frac{1.2.3 \ldots\ldots s}{p+1.p+2 \ldots p+s}\right) \cdot \left(\frac{t.t+1 \ldots t+v-1}{r.r+1 \ldots r+v-1}\right)$$
$$\times \left(1 - \frac{d}{1} \cdot \frac{p+1}{r} + \frac{d.d-1}{1.2} \cdot \frac{p+1.p+2}{r.r+1} - \&c.\right)$$

Thus, taking the same example, and supposing $v = s = 10$, we have $d = r = \frac{1}{2}$, and $r = \frac{21}{2}$; therefore,

$$\int \frac{x^{\prime}}{\sqrt{1-x^{2}}} = \frac{1}{20}\left(\frac{2.4.6 \ldots 20}{1.3.5 \ldots 19}\right).\left(\frac{2.4.6 \ldots 20}{1.3.5 \ldots 19}\right).\left(1 - \frac{1}{2}\cdot\frac{1}{21}\right.$$

$$\left. - \frac{1}{8}\cdot\frac{1.3}{21.23} - \frac{1}{16}\cdot\frac{1.3.5}{21.23.25} - \frac{5}{128}\cdot\frac{1.3.5.7}{21.23.25.27} - \text{\&c.}\right).$$

Hence, if Q denote the length of a quadrant to the radius 1, and v or s be supposed indefinitely great,

$$Q = \frac{2^{2}.4^{2}.6^{2}.8^{2}.10^{2}\ \text{\&c.}}{1^{2}.3^{2}.5^{2}.7^{2}.\ 9^{2}\ \text{\&c.}}$$

It is also evident that $\frac{2^{2}.4}{1^{2}.3^{2}}$, $\frac{2^{2}.4^{2}.6}{1^{2}.3^{2}.5^{2}}$, $\frac{2^{2}.4^{2}.6^{2}.8}{1^{2}.3^{2}.5^{2}.7^{2}}$, &c. form a series, whose terms continually approximate the value of Q.

SECTION XII.

METHOD OF RENDERING THE PRECEDING SERIES MORE CONVERGENT.

From the examples illustrating the preceding theory (in Section X.) it appears, that the complete primitive of the given prime function is expressed by each of two or more infinite series, which differ very much in their rate of convergency. But there is a method of combining two equal series together, so that the com-

pound series shall, by the same number of terms, give a much nearer approximation to the required integral than either of them separately, which method we are now to explain.

Comparing, then, the first and fifth series, and dividing each of them by $\frac{z^p x^{m+1}}{m+1}$, we have,

$$1 - \frac{p}{r+1} \cdot \frac{bx^n}{z} + \frac{p \cdot p-1}{r+1 \cdot r+2} \cdot \frac{b^2x^{2n}}{z^2} - \frac{p \cdot p-1 \cdot p-2}{r+1 \cdot r+2 \cdot r+3} \cdot \frac{b^3x^{3n}}{z^3} + \&c.$$

$$= \frac{z}{a}\left(1 - \frac{q+1}{r+1} \cdot \frac{bx^n}{a} + \frac{q+1 \cdot q+2}{r+1 \cdot r+2} \cdot \frac{b^2x^{2n}}{a^2} - \frac{q+1 \cdot q+2 \cdot q+3}{r+1 \cdot r+2 \cdot r+3} \cdot \frac{b^3x^{3n}}{a^3} + \&c.\right)$$

which series must continue equal, whatever be the values of m, n, p: So that, if for p, q, r, other values of these denoted by p', q', r' be substituted, the equality will still subsist. Now, let v denote a certain number of terms of the former series, then, (putting $p'=p-v$, $r'=r+v$, and $q'=r'+p'=q$,) the remaining terms of the same series, (F being the first of them,) may be thus expressed:

$$\pm F\left(1 - \frac{p'}{r'+1} \cdot \frac{bx^n}{z} + \frac{p' \cdot p'-1}{r'+1 \cdot r'+2} \cdot \frac{b^2x^{2n}}{z^2} - \frac{p' \cdot p'-1 \cdot p'-2}{r'+1 \cdot r'+2 \cdot r'+3} \cdot \frac{b^3x^{3n}}{z^3} + \&c.\right)$$

But this series, from what we observed above, is equal to,

$$\pm \mathrm{F}\frac{z}{a}\left(1 - \frac{q+1}{r'+1}\cdot\frac{bx^n}{a} + \frac{q+1\,.\,q+2}{r'+1\,.\,r'+2}\cdot\frac{b^2x^{2n}}{a^2} - \frac{q+1\,.\,q+2\,.\,q+3}{r'+1\,.\,r'+2\,.\,r'+3}\cdot\frac{b^3x^{3n}}{a^3} + \&c.\right)$$

Consequently,

$$\int z^p x^m x' = \frac{z^p x^{m+1}}{m+1}\left(1 - \frac{p}{r+1}\cdot\frac{bx^n}{z} + \frac{p\,.\,p-1}{r+1.r+2}\cdot\frac{b^2x^{2n}}{z^2} - \&c.\text{ to } v \text{ terms.}\right)$$

$$\pm \mathrm{F}\frac{z^{p+1}x^{m+1}}{a.m+1}\left(1 - \frac{q+1}{r'+1}\cdot\frac{bx^n}{a} + \frac{q+1.q+2}{r'+1.r'+2}\cdot\frac{b^2x^{2n}}{a^2} - \&c.\right)$$

where $r' = r + v$.

In like manner, from the same two series, reversing their order, we deduce this other compound series:

$$\int z^p x^m x' = \frac{z^{p+1}.x^{m+1}}{a.m+1}\left(1 - \frac{q+1}{r+1}\cdot\frac{bx^n}{a} + \frac{q+1.q+2}{r+1.r+2}\cdot\frac{b^2x^{2n}}{a^2} - \ldots (v)\right).$$

$$\pm \mathrm{F}\frac{z^p x^{m+1}}{m+1}\left(1 - \frac{p'}{r'+1}\cdot\frac{bx^n}{z} + \frac{p'.p'-1}{r'+1.r'+2}\cdot\frac{b^2x^{2n}}{z^2} - \&c.\right)$$

EXAMPLE 1.

To approximate $\int \frac{x'}{1+x^2}$, or the arc whose tangent is x.

Here $a = b = 1$, $n = 2$, $p = -1$, $m = o$, $r = \frac{1}{2}$, $q = -\frac{1}{2}$. Suppose $v = 6$; then $r' = r + v = 6\frac{1}{2}$.

Thus,

$$\int \frac{x^r}{1+x^2} = \frac{x}{z} + \frac{2}{3} \cdot \frac{x^3}{z^3} + \frac{2.4}{3.5} \cdot \frac{x^5}{z^5} + \frac{2.4.6}{3.5.7} \cdot \frac{x^7}{z^7} + \frac{2...8}{3...9} \cdot \frac{x^9}{z^9}$$
$$+ \frac{2...10}{3...11} \cdot \frac{x^{11}}{z^{11}} + \frac{2..12}{3..13} \cdot \frac{x^{13}}{z^{13}} \left(1 - \frac{1}{15}x^2 + \frac{1.3}{15.17}x^4\right.$$
$$\left. - \frac{1.3.5}{15.17.19}x^6 + \&c.\right)$$

Hence, if $x = 1$, then $z = 2$, and

$45^\circ = \frac{1}{2} + \frac{1}{3}a + \frac{2}{5}b + \frac{3}{7}c + \frac{4}{9}d + \frac{6}{11}e = .78037517$
$+\frac{12}{13}f - \frac{1}{15}g + \frac{3}{17}h - \frac{6}{19}i + \frac{7}{21}k - \frac{9}{23}l = .00502234$

Thus six terms of each part give 45° ..... $= .78539751$

Or thus,

$$\int \frac{x^r}{1+x^2} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11}$$
$$+ \frac{1}{13}x^{13} \cdot \frac{x}{z}\left(1 + \frac{1}{15}x^2 + \frac{1.2}{15.17}x^4 + \frac{1.2.3}{15.17.19}x^6 + \&c.\right)$$

$45^\circ = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$ ..... $= .74401156$
$+\frac{1}{26} + \frac{1}{15}a + \frac{2}{17}b + \frac{3}{19}c + \frac{4}{21}d + \frac{6}{23}e = .04138596$

Here the same number of terms give $45^\circ = .78539752$

EXAMPLE 2.

To approximate $\int \frac{x'}{1-x^2}$, or $\frac{1}{2}\log.\frac{1+x}{1-x}$.

$$\int \frac{x'}{1-x^2} = \frac{x}{z} - \frac{2}{3}\cdot\frac{x^3}{z^3} + \frac{2.4}{3.5}\cdot\frac{x^5}{z^5} - \frac{2.4.6}{3.5.7}\cdot\frac{x^7}{z^7} + \ldots (v)$$

$$= \mathrm{F}\cdot\frac{z}{1}\left(1 + \frac{1}{15}x^2 + \frac{1.3}{15.17}x^4 + \frac{1.3.5}{15.17.19}x^6 + \&c.\right)$$

Hence, if $x = \frac{1}{3}$, $z = \frac{8}{9}$, and

$\frac{1}{2}$L. $2 = \frac{3}{8} - \frac{1}{12}a + \frac{2}{20}b - \frac{3}{28}c + \frac{4}{36}d - \frac{5}{44}e = .3465731535$

$+\frac{6}{52}f\cdot\frac{8}{9} + \frac{1}{9.15}a'\left(+\frac{3}{9.17}b' + \frac{5}{9.19}c' + \&c.\right) = \quad 4368$

Six terms of the former, and two of the latter $\frac{1}{2}$ L. 2 ..... $= .3465735903$

Or, taking four terms of each series, we have,

$\frac{1}{2}$L. $2 = \frac{3}{8} - \frac{1}{12}a + \frac{2}{20}b - \frac{3}{28}c$ ..... $= .3465401787$

$+\frac{4}{36}d\cdot\frac{8}{9} + \frac{1}{9.11}a' + \frac{3}{9.13}b' + \frac{5}{9.15}c' = \quad 334116$

And the amount is as above, ... $.3465735903$

Or thus, by the other formula:

$$\int \frac{x'}{1-x^2} = x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 + \tfrac{1}{11}x^{11}$$

$$+ \frac{x^{13}}{13}\cdot\frac{1}{z}\left(1 - \frac{1}{15}\frac{x^2}{z} + \frac{1.2}{15.17}\frac{x^4}{z^2} - \frac{1.2.3}{15.17.19}\frac{x^6}{z^3} + \&c.\right.$$

$\frac{1}{2}\text{L.}2=\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\frac{1}{9.3^9}+\frac{1}{11.3^{11}}=.346573536$

$+\frac{1}{13.3^{13}}\cdot\frac{9}{8}\left(-\frac{1}{8\cdot15}a+\frac{1}{8\cdot17}b-\frac{1}{8\cdot19}c+\text{&c.}\right)$ 54

Six terms of the former, and only one of the latter, ... .346573590

SECTION XIII.

ON THE SUMMATION OF SERIES.

In the two following series of numbers,

1, 2, 3, 4, 5 ..... x .....

3, 8, 15, 24, 35 ..... (x^2+2x).....

x and (x^2+2x) are the general terms; that is, x denoting any term of the first series, (x^2+2x) denotes the corresponding term of the second. Thus, if $x=1$, $x^2+2x=3$; if $x=2$, $x^2+2x=8$; &c. Put $y=x^2+2x$; and let Sy denote the sum of the terms of the second series from 3 to y inclusive; then Sy is such a function of x, that, when $x=0$, $Sy=0$; and

since the second series is formed by adding the corresponding terms of these two,

1,	4,	9,	16,	25,	&c.
2,	4,	6,	8,	10,	&c.
3,	8,	15,	24,	35,	&c.

it is plain that $Sy = Sx^3 + S2x$, or $Sx^3 + 2Sx$.

These things premised, let us take the general series, a, b, c, &c. in place of the second, and we shall have,

1,	2,	3,	4,	5	$\overline{x-1}$,	x....
a,	b,	c,	d,	e	v,	y....

In which, let the general term y be any rational and integral function of x, and let it be required to find the sum of the terms from a to y inclusive.

Let v be the term next less than y; then is v the same function of $x - 1$, that y is of x; but when x becomes $x - 1$, y becomes

$$y - y' + \tfrac{1}{2}y'' - \tfrac{1}{6}y''' + \tfrac{1}{24}y^{IV} - \&c.;$$

therefore, $v = y - y' + \frac{1}{2}y'' - \frac{1}{6}y''' + \frac{1}{24}y^{IV} - \&c.$

and $Sv = Sy - Sy' + \frac{1}{2}Sy'' - \frac{1}{6}Sy''' + \frac{1}{24}Sy^{IV} - \&c.$

But $Sv = Sy - y$.

Consequently,

$$\begin{aligned} y &= Sy' - \tfrac{1}{2}Sy'' + \tfrac{1}{6}Sy''' - \tfrac{1}{24}Sy^{IV} + \&c. \\ y' &= Sy'' - \tfrac{1}{2}Sy''' + \tfrac{1}{6}Sy^{IV} - \tfrac{1}{24}Sy^{V} + \&c. \\ y'' &= Sy''' - \tfrac{1}{2}Sy^{IV} + \tfrac{1}{6}Sy^{V} - \tfrac{1}{24}Sy^{VI} + \&c. \\ \&c. & \qquad\qquad\qquad\qquad \&c. \end{aligned}$$

Assume $Sy' = y + \alpha y' + \mathfrak{C} y'' + \gamma y''' + \delta y^{IV} + \&c.$;

then, since

$$\begin{aligned} y &= Sy' - \tfrac{1}{2}Sy'' + \tfrac{1}{6}Sy''' - \tfrac{1}{24}Sy^{IV} + \&c. \\ + \alpha y' &= \quad + \alpha Sy'' - \tfrac{1}{2}\alpha Sy''' + \tfrac{1}{6}\alpha Sy^{IV} - \&c. \\ + \mathfrak{C} y'' &= \quad\quad + \mathfrak{C} Sy''' - \tfrac{1}{2}\mathfrak{C} Sy^{IV} + \&c. \\ + \gamma y''' &= \quad\quad\quad + \gamma Sy^{IV} - \&c. \\ \&c. & \quad\quad\quad\quad \&c. \end{aligned}$$

And the terms on one side of these equations are, by hypothesis, equal to the single term Sy' on the other, the remaining terms on the latter side must destroy one another.

Hence

$$\begin{aligned} \alpha &= \tfrac{1}{2}. \\ \mathfrak{C} &= \tfrac{1}{2}\alpha - \tfrac{1}{6} = \tfrac{1}{12}. \\ \gamma &= \tfrac{1}{2}\mathfrak{C} - \tfrac{1}{6}\alpha + \tfrac{1}{24} = 0. \\ \delta &= \tfrac{1}{2}\gamma - \tfrac{1}{6}\mathfrak{C} + \tfrac{1}{24}\alpha - \tfrac{1}{120} = -\tfrac{1}{720}. \\ \varepsilon &= \tfrac{1}{2}\delta - \tfrac{1}{6}\gamma + \tfrac{1}{24}\mathfrak{C} - \tfrac{1}{120}\alpha + \tfrac{1}{720} = 0. \\ \zeta &= \tfrac{1}{2}\varepsilon - \tfrac{1}{6}\delta + \tfrac{1}{24}\gamma - \tfrac{1}{120}\mathfrak{C} + \tfrac{1}{720}\alpha - \tfrac{1}{5040} = \tfrac{1}{30240}. \\ \&c. & \qquad\qquad \&c. \end{aligned}$$

Now, taking the primitive functions of the terms of the assumed equation, we have,

$$Sy = fy + \alpha y + \beta y' + \gamma y'' + \delta y''' + \varepsilon y^{IV} + \&c.$$

But, since γ, ε, &c. $= 0$, the even derivatives vanish.

Hence $Sy = fy + \frac{1}{2}y + Ay' + By''' + Cy^{V} + \&c.$

The coefficients A, B, C, &c. may be thus determined:

$$A = \tfrac{1}{2}\left|\frac{1}{1.2} - \frac{1}{1.2.3}\right.$$

$$B = \tfrac{1}{2}\left|\frac{1}{1.2.3.4} - \frac{1}{1.2.3}A - \frac{1}{1.2.3.4.5}\right.$$

$$C = \tfrac{1}{2}\left|\frac{1}{1\ldots6} - \frac{1}{1.2.3}B - \frac{1}{1\ldots5}A - \frac{1}{1\ldots7}\right.$$

$$D = \tfrac{1}{2}\left|\frac{1}{1\ldots8} - \frac{1}{1.2.3}C - \frac{1}{1\ldots5}B - \frac{1}{1\ldots7}A - \frac{1}{1\ldots9}\right.$$

&c. &c.

Whence $A = \frac{1}{12}$, $B = -\frac{1}{1\ldots3}\cdot\frac{1}{120}$, $C = \frac{1}{1\ldots5}\cdot\frac{1}{252}$,

$$D = -\frac{1}{1\ldots7}\cdot\frac{1}{240}, \&c.$$

EXAMPLE 1.

For the sum of the series, 1^n, 2^n, 3^n, 4^n, 5^n....x^n..... we have this general expression,

$$Sx^n = \frac{1}{n+1}x^{n+1} + \tfrac{1}{2}x^n + \tfrac{1}{12}\cdot\frac{n}{1}x^{n-1} - \tfrac{1}{120}\cdot\frac{n.n-1.n-2}{1.2.3}x^{n-3}$$

$$+ \tfrac{1}{252}\cdot\frac{n\ldots n-4}{1\ldots5}x^{n-5} - \tfrac{1}{240}\cdot\frac{n\ldots n-6}{1\ldots7}x^{n-7} + \tfrac{1}{132}\ \&c.$$

Hence

$$\begin{aligned}
Sx^0 &= x.\\
Sx &= \tfrac{1}{2}x^2 + \tfrac{1}{2}x.\\
Sx^2 &= \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x.\\
Sx^3 &= \tfrac{1}{4}x^4 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2.\\
Sx^4 &= \tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x.\\
Sx^5 &= \tfrac{1}{6}x^6 + \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2.\\
&\text{\&c.} \qquad\qquad \text{\&c.}
\end{aligned}$$

EXAMPLE 2.

Required the sum of the series, 1, 27, 125, 343, 729, &c. whereof the general term is $(2x-1)^3$.

$$\begin{aligned}
\text{Here } y &= 8x^3 - 12x^2 + 6x - 1.\\
\text{Hence } \int y &= 2x^4 - 4x^3 + 3x^2 - x.\\
y' &= 24x^2 - 24x + 6.\\
y'' &= 48x - 24.\\
y''' &= 48.\\
y^{IV} &= 0.\\
&\text{\&c.} \quad \text{\&c.}
\end{aligned}$$

$$\begin{array}{rl}
\int y = & 2x^4 - 4x^3 + 3x^2 - x.\\
+\tfrac{1}{2}y = & \quad + 4x^3 - 6x^2 + 3x - \tfrac{1}{2}.\\
+\tfrac{1}{12}y' = & \quad\quad + 2x^2 - 2x + \tfrac{1}{2}.\\
-\tfrac{1}{720}y''' = & \quad\quad\quad - \tfrac{1}{15}.\\
\hline
Sy = & 2x^4 - x^2 - \tfrac{1}{15}.
\end{array}$$

But, when $x = 0$, $Sy = 0$; therefore $Sy = 2x^4 - x^2$.

Or thus,

$$Sy = 8Sx^3 - 12Sx^2 + 6Sx - 1Sx^0.$$

Now $8 . Sx^3 = 2x^4 + 4x^3 + 2x^2$. See example 1.

$-12 . Sx^2 = \quad -4x^3 - 6x^2 - 2x.$

$+\ 6 . Sx = \quad +3x^2 + 3x.$

$-\ 1 . Sx^0 = \quad -x.$

Therefore, $Sy = 2x^4 - x^2$, as before.

The sum of a series of reciprocals, whereof $\frac{1}{y}$ is the general term, may be easily obtained, provided y be such a function of x, as to be composed of two or more binomial factors, of which 1 is the common difference.

Thus, let $\frac{1}{y} = \frac{1}{(a+x) . (a+x+1)}$, and suppose

$$S = \frac{1}{a+1} + \frac{1}{a+2} + \frac{1}{a+3} + \ldots + \frac{1}{a+x};$$

then

$$S - \frac{1}{a+1} + \frac{1}{a+x+1} = \frac{1}{a+2} + \frac{1}{a+3} + \frac{1}{a+4} + \ldots\ldots + \frac{1}{a+x+1}.$$

And, by subtraction,

$$\frac{1}{a+1} - \frac{1}{a+x+1} = \frac{1}{(a+1)(a+2)} + \frac{1}{a+2 \,.\, a+3} + \ldots\ldots + \frac{1}{a+x \,.\, a+x+1}.$$

Hence

$$\mathrm{S} \,.\, \frac{1}{a+x \,.\, a+x+1} = \frac{1}{a+1} - \frac{1}{a+x+1}.$$

In like manner,

$$\mathrm{S} \,.\, \frac{1}{a+x \,.\, a+x+1 \,.\, a+x+2} = \tfrac{1}{2} \,.\, \frac{1}{a+1 \,.\, a+2} - \tfrac{1}{2} \,.\, \frac{1}{a+x+1 \,.\, a+x+2}.$$

$$\mathrm{S} \,.\, \frac{1}{a+x \,.\, a+x+1 \,.\, a+x+2 \,.\, a+x+3} = \tfrac{1}{3} \,.\, \frac{1}{a+1 \,.\, a+2 \,.\, a+3} - \tfrac{1}{3} \,.\, \frac{1}{a+x+1 \,.\, a+x+2 \,.\, a+x+3}.$$

&c. &c.

Thus, with respect to the figurate numbers, 1, 3, 6, 10, &c. 1, 4, 10, 20, &c. &c. the sums of their reciprocals will be as follows:

$$\mathrm{S} \,.\, \frac{1.2}{x \,.\, x+1} = \tfrac{2}{1} - \frac{2}{x+1}.$$

$$\mathrm{S} \,.\, \frac{1.2.3}{x \,.\, x+1 \,.\, x+2} = \tfrac{3}{2} - \frac{1.3}{x+1 \,.\, x+2}.$$

$$\text{S}.\frac{1.2.3.4}{x.x+1.x+2.x+3} = \frac{4}{3} - \frac{1.2.4}{x+1.x+2.x+3}.$$

$$\text{S}.\frac{1.2.3.4.5}{x.x+1.x+2.x+3.x+4} = \frac{5}{4} - \frac{1.2.3.5}{x+1.x+2.x+3.x+4}.$$

&c. &c.

EXAMPLE.

To find the sum of the series, $\frac{1}{3}$, $\frac{1}{21}$, $\frac{1}{45}$, $\frac{1}{77}$, &c. whereof $\frac{1}{4x^2+4x-3} = \frac{1}{2x-1.2x+3}$ is the general term.

Here $\frac{1}{2x-1.2x+3} = \frac{1}{4}\cdot\frac{1}{2x-1} - \frac{1}{4}\cdot\frac{1}{2x+3}$

$$= \frac{1}{8}\cdot\frac{1}{x-\frac{1}{2}} - \frac{1}{8}\cdot\frac{1}{x+\frac{3}{2}}.$$

But $\text{S}.\frac{1}{x-\frac{1}{2}.x+\frac{1}{2}}$, or $\text{S}.\frac{1}{x-\frac{1}{2}} - \text{S}.\frac{1}{x+\frac{1}{2}} = \frac{2}{1} - \frac{2}{2x+1}$,

and $\text{S}.\frac{1}{x+\frac{1}{2}} - \text{S}.\frac{1}{x+\frac{3}{2}} = \frac{2}{3} - \frac{2}{2x+3}$;

therefore $\text{S}.\frac{1}{x-\frac{1}{2}} - \text{S}.\frac{1}{x+\frac{3}{2}} = \frac{8}{3} - \frac{8x+8}{2x+1.2x+3}$.

Consequently, $\text{S}.\frac{1}{4x^2+4x-3} = \frac{1}{3} - \frac{x+1}{2x+1.2x+3}$.

The following examples point out a method of investigating the sum in certain cases.

EXAMPLE 1.

To find the sum of the series,

$$x + 2x^2 + 3x^3 + 4x^4 \ldots + nx^n.$$

Put $s = x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots + nx^n$; then

$$\frac{sx'}{x} = x' + 2xx' + 3x^2x' + 4x^3x' + \ldots + nx^{n-1}x',$$

and $$\int\frac{sx'}{x} = x + x^2 + x^3 + x^4 \ldots\ldots + x^n = \frac{x - x^{n+1}}{1 - x}.$$

Hence, taking the prime functions,

$$\frac{sx'}{x} = \frac{x' - \overline{n+1}\,x^nx'}{1-x} + \frac{xx' - x^{n+1}x'}{(1-x)^2}.$$

Consequently, $$s = \frac{x - \overline{n+1}\,.\,x^{n+1} + nx^{n+2}}{(1-x)^2}.$$

EXAMPLE 2.

To find the sum of this general series,

$$ax^m + \overline{a+d}\,.\,x^{m+r} + \overline{a+2d}\,.\,x^{m+2r} + \ldots\ldots$$
$$+ \overline{a + \overline{n-1}\,.\,d}\,.\,x^{m+\overline{n-1}.r}.$$

This may be summed in a similar manner, the proper multiplier of s being determined thus:

Suppose it to be px^tx'; then

$$\int psx^tx' = apx^{m+t}x' + \overline{a+d}.p.x^{m+r+t}x' + \overline{a+2d}.p.x^{m+2r+t}x'$$
$$+ \ldots\ldots\ldots + \overline{a + \overline{n-1}\,.\,d}\,.\,p\,.\,x^{m+\overline{n-1}.r+t}x'.$$

Now this series is integrable, if the coefficient in each term, diminished by 1, be equal to the index of x in the same term; that is,

$$\text{if } \overline{a+n-1}\,.\,d\,.\,p-1 = m+\overline{n-1}\,.\,r+t.$$

This equation, as there are two indeterminate quantities, p and t, may be resolved into

$$ap-1 = m+t, \text{ and } pd = r.$$

$$\text{Whence } p = \frac{r}{d}, \text{ and } t = \frac{ar}{d} - m - 1.$$

Thus, putting $\frac{ar}{d} = q$, we have,

$$\frac{r}{d}\int sx^tx^v = x^q + x^{q+r} + x^{q+2r} + x^{q+3r} + \ldots + x^{q+nr}$$

$$= \frac{x^q - x^{q+nr}}{1-x^r};$$

from which, by taking the prime functions on each side, we may find the value of s.

EXAMPLE 3.

To find the sum of

$$\frac{1}{2.3} + \frac{2}{2^3.5} + \frac{6}{2^5.7} + \frac{20}{2^7.9} + \frac{70}{2^9.11} + \frac{252}{2^{11}.13} + \&c.$$

This series resembling that for a quadrant, in terms of its sine; let x be the sine of the arc z; then

$$z = x + \frac{1}{2.3}x^3 + \frac{3}{2^3.5}x^5 + \frac{10}{2^5.7}x^7 + \frac{35}{2^7.9}x^9 + \frac{126}{2^9.11}x^{11} + \&c.$$

$$z' = x' + \tfrac{1}{2}x^2x' + \tfrac{3}{2^3}x^4x' + \tfrac{10}{2^5}x^6x' + \tfrac{35}{2^7}x^8x' + \&c.$$

$$\tfrac{1}{2}x^2z' = \tfrac{1}{2}x^2x' + \tfrac{1}{2^2}x^4x' + \tfrac{3}{2^4}x^6x' + \tfrac{10}{2^6}x^8x' + \&c.$$

and

$$\tfrac{1}{2}\int x^2z' = \tfrac{1}{2.3}x^3 + \tfrac{2}{2^3.5}x^5 + \tfrac{6}{2^5.7}x^7 + \tfrac{20}{2^7.9}x^9 + \&c.$$

which, when $x = 1$, is the given series. But, in that case,

$\tfrac{1}{2}\int x^2z'$, or $\tfrac{1}{2}\int \frac{x^2x'}{\sqrt{1-x^2}} = \tfrac{1}{4}Q$. (See ex. 1. Sect. XI.)

Thus the sum of the given series is equal to an arc of $22\frac{1}{2}^\circ$ to the radius 1.

SECTION XIV.

SOLUTION OF DERIVATIVE EQUATIONS OF THE FIRST ORDER.

THE methods of resolving derivative equations, as well as the manner in which they become complicated with their primitives, and thereby more difficult to be resolved, will be best explained by examples.

EXAMPLE 1.

$$x' = ay'.$$

This is one of the most simple that can be adduced, where the primitive equation appears at once to be $x = ay$. But if it be known that x becomes equal to some constant quantity c, when $y = 0$, the primitive, in that case, will be $x = ay + 0$.

EXAMPLE 2.

$$yx' = xy'.$$

Here, by separating x and y, we get $\frac{x'}{x} = \frac{y'}{y}$, the primitive of which is evidently $l.x = l.y$, or $l.x = l.y + l.a$. Whence $x = ay$.

Or thus, since $yx' - xy' = 0$, $\frac{yx' - xy'}{y^2} = 0$. Now the primitive of one side of this equation is known to be $\frac{x}{y}$, and the primitive of the other to be any constant quantity a; therefore $\frac{x}{y} = a$, and $x = ay$, as before. This second example is compounded of the first and its primitive,

$$\begin{array}{l} x' = ay' \\ ay = x \\ \hline yx' = xy'. \end{array}$$

EXAMPLE 3.

$$xx' - axy' - ayx' + a^2yy' = 0.$$

The primitive here is evidently $\frac{1}{2}x^2 - axy + \frac{1}{2}a^2y^2 = 0$, (or $= b$). Hence $x - ay = 0$, (or $= \sqrt{2b}$).

This example is also compounded of the first, $x' - ay' = 0$, and its primitive, $x - ay = 0$.

EXAMPLE 4.

$$yxx' - cy^2x' - x^2y' + cxyy' = 0.$$

It is easy to see that this equation is divisible by $x - cy$; therefore $x - cy = 0$.

Or thus, putting $y = xz$, we have,

$$zx^2x' - cz^2x^2x' - x^2z' - zx^2x + cx^2zz' + cz^2x^2x' = 0,$$

$$\text{or, } -x^2z' + cx^2zz' = 0.$$

$$\text{Hence } -1 + cz = 0, \text{ or } z = \frac{1}{c}.$$

Consequently, $cy = x$, as before.

This example is compounded of the second and its primitive.

EXAMPLE 5.

$$x'\sqrt{x^2 + y^2} = yx' - xy'.$$

Putting $y = xz$, we have $y' = xz' + zx'$, and $x'\sqrt{x^2 + x^2z^2} = zxx' - x^2z - zxx'$, or $x'\sqrt{1 + z^2} = -xz'$.

Hence $\frac{z'}{\sqrt{1+z^2}} + \frac{x'}{x} = 0$; of which the primitive is,

$$l.(z + \sqrt{1+z^2}) + l.x = l.a,$$

$$\text{or } l.\frac{y+\sqrt{x^2+y^2}}{x} + l.x = l.a;$$

therefore $y + \sqrt{x^2+y^2} = a$, or $x^2 = a^2 - 2ay$.

This equation, compounded with its first derivative, will give the above derivative equation; thus,

Equation, $x^2 = a^2 - 2ay$. Hence $\sqrt{x^2+y^2} = (a-y)$.

1st Der. $-ay' = xx'$

$$-xy' = ax' - 2yx', \text{ or } x'(a-y) = yx' - xy'.$$

Therefore $x'\sqrt{x^2+y^2} = x'(a-y) = yx' - xy'$.

EXAMPLE 6.

$$y + \frac{y'(a-x)}{x'} = x - \frac{yx'}{y'}.$$

By putting $x' = 1$, this equation becomes,

$$y + ay' - xy' = x - \frac{y}{y'};$$

of which the prime function is,

$$y' + ay'' - xy'' - y' = 1 - 1 + \frac{yy''}{y'^2}.$$

Hence $a - x = \frac{y}{y'^2}$, or $\frac{y'^2}{y} = \frac{1}{a-x} = \frac{x'^2}{a-x}$, and

$y'y^{-\frac{1}{2}} = x'(a-x)^{-\frac{1}{2}}$, whose primitive, $2y^{\frac{1}{2}} = -2(a-x)^{\frac{1}{2}}$, or $\sqrt{y} = c - \sqrt{a-x}$.

Otherwise,

If we consider x as a function of y, and put $y' = 1$, we have,

$$yx' + a - x = xx' - yx'^2;$$

of which the prime function is,

$$yx'' = xx'' - 2yx'x'',$$

(omitting the terms which destroy one another,) or $y = x - 2yx'$. Hence $yx' = xx' - 2yx'^2$, which, subtracted from the first equation, leaves $a-x = yx'^2$; therefore, $\frac{x'^2}{a-x} = \frac{1}{y} = \frac{y'^2}{y}$, as before.

EXAMPLE 7.

$$xx' + yy' = 2yx'.$$

Put $y = x - v$; then $xx' + xx' - vx' - xv' + vv' = 2xx' - 2vx'$; that is, $vx' - xv' + vv' = 0$.

Hence $\frac{vx' - xv'}{v^2} = -\frac{v'}{v}$, of which the primitive is,

$$\frac{x}{v} = l \cdot \frac{1}{v} + l \cdot c.$$

Therefore $\epsilon^{\frac{x}{v}} = \frac{c}{v}$, and $(x-y)\epsilon^{\frac{x}{x-y}} = c$.

EXAMPLE 8.

$$az'y'^2 = y(z'^2 + y'^2)^2.$$

By putting $y' = 1$, and $z'^2 + 1 = v^2$, we have $z'z'' = vv'$, and $az' = v^4y$; therefore $y = \frac{az'}{v^4}$, and y' or $1 = \frac{az''}{v^4} - \frac{4az'v'}{v^5}$. Hence $z' = \frac{az'z''}{v^4} - \frac{4az'^2v'}{v^5}$; but $z'z'' = vv'$, and $z'^2 = v^2 - 1$; therefore $z' = 4av^{-5}v' - 3av^{-3}v'$; of which the primitive is $z = \frac{3av^2 - 2a}{2v^4}$. But when $z' = 0$, $v = 1$, and the last expression becomes $\frac{a}{2}$; therefore the correct primitive is $z = \frac{3av^2-2a}{2v^4} - \frac{a}{2}$. From which equation v may be found in terms of z; then $\sqrt{v^2 - 1}$ being substituted for z in the equation $y = \frac{az'}{v^4}$, we shall have y in terms of z.

EXAMPLE 9.

$$x'\sqrt{a + by + cy^2} = y'\sqrt{a + bx + cx^2}.$$

Supposing x, y to be such functions of some third quantity,

$$\text{that } x' = \sqrt{a + bx + cx^2},$$
$$\text{then } y' = \sqrt{a + by + cy^2}.$$

Therefore, $x'^2 = a + bx + cx^2$, and $y'^2 = a + by + cy^2$.

Hence, $2x' = b + 2cx$, and $2y' = b + 2cy$.

Put $x + y = p$, and $x - y = q$;

then $x' + y' = p'$, and $x'^2 - y'^2 = p'q'$, &c.

And $p'q' = bq + cpq$.

$$p'' = b + cp = \frac{p'q'}{q}, \text{ or } \frac{p''}{p'} = \frac{q'}{q}, \text{ of which}$$

the primitive is $l \cdot p' = l \cdot q + l \cdot \mathrm{A}$, or $p' = \mathrm{A}q$, that is, $\sqrt{a + bx + cx^2} + \sqrt{a + by + cy^2} = \mathrm{A}(x - y)$.

EXAMPLE 10.

$$\frac{x'}{\sqrt{a + bx + cx^2 + dx^3 + ex^4}} = \frac{y'}{\sqrt{a + by + cy^2 + dy^3 + ey^4}}.$$

Suppose $x' = \sqrt{a + bx + cx^2 + dx^3 + ex^4}$,

then $y' = \sqrt{a + by + cy^2 + dy^3 + ey^4}$.

And proceeding as above,

$$x'^2 = a + bx + cx^2 + dx^3 + ex^4, \text{ and}$$

$$y'^2 = a + by + cy^2 + dy^3 + ey^4.$$

Put $x + y = p$, &c.

Hence, $2x'' = b + 2cx + 3dx^2 + 4ex^3$.

$$2y'' = b + 2cy + 3dy^2 + 4ey^3.$$

$$p'' = b + cp + \frac{3d}{4}(p^2 + q^2) + \tfrac{1}{2}e(p^3 + 3pq^2).$$

.....

Or, $p''q = bq + cpq + \frac{d}{4}(3p^2q + 3q^3) + \frac{1}{2}e(p^3q + 3pq^2)$.

But $p'q' = bq + cpq + \frac{d}{4}(3p^2q + q^3) + \frac{1}{2}e(p^3q + pq^3)$.

Therefore, $p''q - p'q' = \frac{1}{2}dq^3 + epq^3$, and multiplied by $\frac{2p'}{q^3}$,

$\frac{2p'p''}{q^3} - \frac{2p'^2q'}{q^3} = dp' + 2epp'$, of which the primitive is $\frac{p'^2}{q^3} = dp + ep^2 + A^2$, hence $p' = q\sqrt{A^2 + dp + ep^2}$, that is,

$$\sqrt{a + bx + cx^2 + dx^3 + ex^4} + \sqrt{a + by + cy^2 + dy^3 + ey^4} = (x - y)\sqrt{A^2 + d(x + y) + e(x + y)^2}.$$

If the above value of p' be substituted in the equation $p'q' = bq + cpq +$ &c., we shall have,

$$q' = \frac{b + cp + \frac{1}{4}d(3p^2 + q^2) + \frac{1}{2}e(p^3 + pq^2)}{\sqrt{A^2 + dp + ep^2}} = \frac{M}{\sqrt{N}} \text{ suppose.}$$

Hence, (putting $\sqrt{P}$, $\sqrt{Q}$ for the radical quantities in the given equation,) we have these primitive equations:

$$\sqrt{P} - \sqrt{Q} = \frac{M}{\sqrt{N}},$$
$$\sqrt{P} + \sqrt{Q} = q\sqrt{N}.$$

And $P + Q = qM$, an equation free from surds.

EXAMPLE 11.

$$x'\sqrt{m+ny+cy^2} - y'\sqrt{a+bx+cx^2} = 0.$$

Put v, z for the radical quantities, then $vx' = zy'$; also, $v^2 = m + ny + cy^2$, and $z^2 = a + bx + cx^2$.

Hence, $2vv' = ny' + 2cyy'$, and

$bx' + 2cxx' = 2zz'$, their product divided by $2vx' = 2zy'$ is $bv' + 2cxv' = nz' + 2cyz'$.

But $2cvx' = 2czy'$; therefore,

$bv' + 2cvx' + 2cxv' = nz' + 2cyz' + 2czy'$, of which the primitive is $bv + 2cvx = A + nz + 2czy$, that is,

$$(b+2cx)\,.\,\sqrt{m+ny+cy^2} = A+(n+2cy)\sqrt{a+bx+cx^2}.$$

Otherwise.

Supposing $x'=1$, $m+ny+cy^2=y'^2(a+bx+cx^2)$, and by derivation, $n+2cy = y'(b+2cx)+2y''(a+bx+cx^2)$.

Also, $2cy' = y'.2c+3y''(b+2cx)+2y'''.(a+bx+cx^2)$.

Hence, $\dfrac{2y'''}{y''} + \dfrac{3(b+2cx)}{a+bx+cx^2} = 0$, of which the primitive is $2l\,.\,y'' + 3l\,.\,(a + bx + cx^2) = 2l\,.\,A$. Thus $y'' = A(a + bx + cx^2)^{-\frac{1}{3}}$. This value of y'', and the value of y' (from the first equation,) being substituted in the second equation, we get,

$$(n+2cy).\sqrt{a+bx+cx^2} = (b+2cx).\sqrt{m+ny+cy^2}+2A.$$

EXAMPLE 12.

Given $y' + Ay = B$, where A and B denote given functions of x.

Suppose $v' = Av = 0$, then $-\frac{v'}{v} = A$, and $l.\frac{1}{v} = \int A$.

Next, suppose $vz = y$, then $vz' + zv' + Avz = B$. But, by the first hypothesis, $z\ (v' + Av) = 0$; therefore $vz' = B$; and $z' = \int \frac{B}{v}$; whence y may be found.

Thus, if $ay' - y = cx^2$, or $y' - \frac{1}{a} \cdot y = \frac{c}{a} \cdot x^2$.

Here, $-\frac{v'}{v} = -\frac{1}{a}$, $l.v = \frac{x}{a}$, and $v = \epsilon^{\frac{x}{a}}$. But $B = \frac{c}{a}x^2$, therefore, $z' = \frac{c}{a}x^2.\epsilon^{-\frac{x}{a}}$. Suppose $z = (Ex^2 + Fx + G)\epsilon^{-\frac{x}{a}}$, then $z' = (2Ex + F)\epsilon^{-\frac{x}{a}} - \left(\frac{E}{a}x^2 + \frac{F}{a}x + \frac{G}{a}\right)\epsilon^{-\frac{x}{a}}$.

Hence, equating these values of z', and supposing $\frac{c}{a}x^2 = -\frac{E}{a}x^2$, we have $E = -c$, also $2Ex - \frac{F}{a}x = 0$, and $F - \frac{G}{a} = 0$; whence $F = -2ac$, and $G = -2a^2c$.

Therefore, $z = Q - (cx^2 + 2acx + 2a^2c)\epsilon^{-\frac{x}{a}}$.

But $v = \epsilon^{-\frac{x}{a}}$.

Consequently $y = Q\epsilon^{\frac{x}{a}} - (cx^2 + 2acx + 2a^2c)$.

In general, it is necessary to approximate the primitive by an infinite series; and even by such a method the exact primitive may sometimes be determined, as in the next example.

EXAMPLE 13.

Given $y - ay' + cx^2 = 0$.

Assume $y = 1 + Ax + Bx^2 + Cx^3 + \&c.$

then $-ay' = -Aa - 2aBx - 3aCx^2 - 4aDx^3 - \&c.$

$+ cx^2 = \qquad\qquad + cx^2.$

$= 0.$

Whence, $A = \frac{1}{a}$, $B = \frac{1}{2a^2}$, $C = \frac{1+2a^2c}{2.3.a^3}$, $D = \frac{1+2a^2c}{2.3.4.a^4}$ &c.

And $y = 1 + \frac{x}{a} + \frac{x^2}{2a^2} + \frac{1+2a^2c}{2.3.a^3}x^3 + \frac{1+2a^2c}{2.3.4.a^4}x^4 + \&c.$

But since $\epsilon^{\frac{x}{a}} = 1 + \frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{2.3.a^3} + \frac{x^4}{2.3.4.a^4} + \&c.$

therefore $y = \epsilon^{\frac{x}{a}} + 2a^2c\left(\epsilon^{\frac{x}{a}} - 1 - \frac{x}{a} - \frac{x^2}{2a^2}\right)$;

or $y = (1 + 2a^2c)\,\epsilon^{\frac{x}{a}} - (2a^2c + 2acx + cx^2)$.

EXAMPLE 14.

Given $\frac{n-1-mx}{x(1-x)}y + y' = \frac{n-1}{x(1-x)}$.

The given equation (putting $n-1=r$) being reduced to $\overline{r-mx}.y + \overline{x-x^2}.y' - r = 0$.

Assume $y = A + Bx + Cx^2 + Dx^3 + \&c.$ then

$$\begin{aligned} ry &= rA + rBx + rCx^2 + rDx^3 + \&c. \\ -mxy &= -mAx - mBx^2 - mCx^3 - \&c. \\ +xy' &= +Bx + 2Cx^2 + 3Dx^3 + \&c. \\ -x^2y' &= -Bx^2 - 2Cx^3 - \&c. \\ -r &= -r. \\ \hline = 0. & \end{aligned}$$

Whence, by equating the coefficients of the same power to 0, we get $A=1$, $B=\frac{m}{r+1}$, $C=\frac{m.m+1}{r+1.r+2}$,

$$D=\frac{m.m+1.m+2}{r+1.r+2.r+3}, \&c.$$

Therefore $y = 1 + \frac{m}{n}x + \frac{m.m+1}{n.n+1}x^2 + \frac{m.m+1.m+2}{n.n+1.n+2}x^3 + \&c.$

Otherwise,

Since $\overline{n-1-mx}.y+x(1-x).y' = n-1$, put $\overline{n-1} = r$, $\overline{m-n} = q$,

then $r(1-x)y - \overline{q+1}.xy + x(1-x).y' = r$, and, multiplying the whole by $x^{r-1}(1-x)^q$, we get,

$$r.x^{r-1}(1-x)^{q+1}.y - \overline{q+1}.x^r(1-x)^q y + x^r(1-x)^{q+1}.y'$$

$$= (rx^{r-1}(1-x)^q =) = rx^{r-1} - \frac{r.q}{1}x^r + \frac{rq.q-1}{1.2}x^{r+1}$$

$$- \frac{rq.q-1.q-2}{1.2.3}x^{r+2} + \&c.$$

And, taking the primitive of each side, we have,

$$x^r(1-x)^{q+1}.y = x^r - \frac{r.q}{r+1}x^{r+1} + \frac{r.q.q-1}{1.2.r+2}x^{r+2}$$

$$- \frac{r.q.q-1.q-2}{1.2.3.r+3}x^{r+3} + \&c.;$$

hence,

$$y = \frac{1 - \frac{rq}{r+1}x + \frac{r.q.q-1}{1.2.r+2}x^2 - \frac{r.q.q-1.q-2}{1.2.3.r+3}x^3 + \&c.}{(1-x)^{q+1}}$$

EXAMPLE 15.

To find the true anomaly, in terms of the eccentric, from this equation :

$$t' + c\cos.x.t' - \sqrt{1-c^2} = 0.$$

EXAMPLE

Given $\frac{n-1-mx}{x(1-x)}y$

The given equation (pu

duced to $\overline{r-mx}\,.\,y$ +

Assume $y = A + Bx$ +

$$ry = rA + rBx$$
$$-mxy = -mA$$
$$+xy' = +B$$
$$-x^2y =$$
$$-r = -r.$$
$$= 0.$$

Whence, by equ

to 0, we get A

D

Therefo

$$\ldots 69c + 360c^2)}{(1+c)^{11}} m^7 - \frac{2bc}{1 \ldots 9}$$

Since $n-1-$ ...

$$\ldots 273c^2 - 21960c^3}{(1+c)^{14}} m^9 + \&c.$$

then $r(1-x)y-$ ...

multiplying the ... than 90°, c becomes negative, and

$r.x^{n-1}(1-x)^{n+1}y-$ ...

$$= (rx^n(1-x^n = \ldots - \tfrac{1}{6} \cdot \frac{2bc}{(1-c)^5} m^3 + \tfrac{1}{120} \cdot \frac{2bc(1+12c)}{(1-c)^8} m^5$$

$$\ldots \frac{2bc(1+69c+360c^2)}{(1-c)^{11}} m^7 + \frac{2bc}{1 \ldots 9}$$

And, taking ...

$$\ldots 6c^2 + 6273c^2 + 21960c^3}{(1-c)^{14}} m^9 + \&c.$$

$e^{(1-x)^n}$ ... ier series converges quickly when the planet ... aphelion; and the latter, when it is near ... elion.

hence ...

$y =$ ... it be required to find the true anomaly of Mer- ... when the mean anomaly is 45°.

... n the orbit of Mercury, (by La Lande's Astronomy,)

...g. c = 9.3128388.	Log. $(1+c)$ = 0.0811718.
...og. b = 9.9906293.	Log. $(1-c)$ = 9.9000809.

When the mean distance = 1.

Now, putting $t = Am + Bm^3 + Cm^5 +$ &c. let the logarithms of the coefficients A, B, C, &c. be computed according to the above formulæ, and reduced to the logarithms of seconds, (by adding 5.3144251 to each); and they will be as follows:

t less than 90°.	t greater than 90°.
Log. A = 5.1427108.	+ Log. A = 5.5048806.
B = 3.7349130.	— B = 4.6403375.
C = 2.3565469.	+ C = 4.1788946.
D = 0.6298992.	— D = 3.7981982.
E = 9.3177534.	+ E = 2.0751605.

The logarithm of 45° (in parts of the radius) being 9.8950899, (= L suppose,) we have,

L. 9.8950899	3L. 9.6852697	5L. 9.4754495	7L. 9.2856
L. A 5.1427108	L. B 3.7349130	L. C 2.3565469	L. D 0.6299

L. a 5.0378007, L. b 3.4201827, L. c 1.8319964, L. d 9.9155

Then a = 109093″.9
b = 2631 .4
c = 67 .9
d = .8

t = 111794″.0 = 31° 3′ 14″.

Otherwise,

It has been shewn (in Section VII.) that

$$(1 - c \cos. t)^{-s} = \frac{1}{s} + \frac{1}{s}\left(\frac{n+1}{2^{n-1}} c^n + \frac{n \cdot n-1}{2^{n+1}} c^{n+2} + \frac{n \cdot n-1 \cdot n+1}{1 \cdot 2 \cdot 2^{n+3}} c^{n+4} + \&c.\right);$$

where $s = (1 - c^2)^{\frac{1}{2}}$, and n is to be put $= 1, 2, 3,$ &c. successively. Hence

$$m = t + \left(\frac{n+1}{n \cdot 2^{n-1}} c^n + \frac{n-1}{2^{n+1}} c^{n+2} + \frac{n-1 \cdot n+1}{1 \cdot 2 \cdot 2^{n+3}} c^{n+4} + \frac{n-1 \cdot n+1 \cdot n+4}{1 \cdot 2 \cdot 3 \cdot 2^{n+5}} c^{n+6} + \&c.\right) \sin. nt.$$

From this general coefficient it is easy to deduce the law by which a particular coefficient may be derived from the two which immediately precede it. Thus, $p \cdot \sin. \overline{n-1} \cdot t$, $q \cdot \sin. nt$, $r \cdot \sin. \overline{n+1} \cdot t$, being three adjacent terms of the series, we find

$$r = \frac{2n^2}{(n^2-1)c} \cdot q - p.$$

And since the coefficient of sin. t is $2c$, and that of sin. $2t = \frac{2s - 2 + 3c^2}{c^3}$, each of the coefficients may be expressed in finite terms.

We are next to find t by a similar series, in terms of m; that is, to find

$$t = m + \alpha \sin. m + \mathfrak{C} \sin. 2m + \gamma \sin. 3m + \&c.;$$

for which purpose, let the sines of m and its multiples be expressed in terms of the arc, and we shall have,

$$t = (1 + \alpha + 2\mathfrak{C} + 3\gamma + 4\delta + \&c.)\, m - \tfrac{1}{6}\cdot(\alpha + 2^3\mathfrak{C} + 3^3\gamma + 4^3\delta + \&c.)\, m^3 + \tfrac{1}{120}(\alpha + 2^5\mathfrak{C} + 3^5\gamma + 4^5\delta + \&c.)\, m^5 - \tfrac{1}{5040}(\alpha + 2^7\mathfrak{C} + 3^7\gamma + \&c.)\, m^7 + \&c.$$

But, in the preceding investigation, we found

$$t = \frac{(1-c^2)^{\frac{1}{2}}}{(1+c)^2}\, m + \tfrac{1}{6}\cdot\frac{2c\,(1-c^2)^{\frac{1}{2}}}{(1+c)^3}\, m^3 - \tfrac{1}{120}\cdot\frac{2c\,(1-12c)\,(1-c^2)^{\frac{1}{2}}}{(1+c)^5} + \&c.$$

$$= (1+A)m - \tfrac{1}{6}\cdot B m^3 + \tfrac{1}{120}\cdot C m^5 - \tfrac{1}{5040}\cdot D m^7 + \&c.$$

So that A, B, C, &c. are given, to find α, $\mathfrak{C}$, γ, &c.; and, first,

$$\alpha = -A - \tfrac{1}{2}B - \tfrac{3}{8}C - \tfrac{5}{16}D - \&c.$$

$$\mathfrak{C} = -\tfrac{1}{2}A - \tfrac{1}{2}B - \tfrac{5}{8}C - \tfrac{15}{16}D - \tfrac{175}{64}c^4 + \&c.$$

$$\gamma = -\tfrac{1}{3}A - \tfrac{2}{3}B - \tfrac{5}{6}C - \tfrac{35}{36}D - \tfrac{1225}{288}c^4 - \tfrac{13631}{64}c^5 + \&c.$$

&c.

Next, $\alpha + 2\mathfrak{C} + 3\gamma + \&c. = A.$

$\alpha + 2^2\mathfrak{C} + 3^2\gamma + \&c. = B.$

$\alpha + 2^3\mathfrak{C} + 3^3\gamma + \&c. = C.$

&c. &c.

Hence

$(2^2-1)\alpha - 3(3^2-2^2)\gamma - \&c. = 2^2.A - B.$

$(2^2-1)\alpha - 3^3(3^2-2^2)\gamma - \&c. = 2^2.B - C.$

$(3^2-1).(2^2-1)\alpha + \&c. \ldots = 3^2.2^2.A - (3^2+2^2)B + C.$

And $\alpha = \frac{3}{2}A - \frac{13}{24}B + \frac{1}{24}C.$

Thus, $\frac{3}{2}A = -3c + \frac{15}{4}c^3 - \frac{9}{2}c^5.$

$-\frac{13}{24}B = +\frac{13}{12}c - \frac{65}{12}c^3 + \frac{377}{24}c^5.$

$+\frac{1}{24}C = -\frac{1}{12}c + \frac{20}{12}c^3 - \frac{263}{24}c^5.$

$= \alpha = -2c \ldots * \ldots + \frac{1}{4}c^5.$

To find α more accurately, so as to extend to the ninth power of c, for example, we must employ more of the preceding equations, and carry the series for A, B, C, &c. to the same power of c. The law of continuation is easily discovered.

But if it be observed that the values of α, and the other coefficients of the sines of the *odd* multiples of m consist of the odd powers only of c; and the values

of ϵ, and the other coefficients of the sines of the *even* multiples of m consist of the even powers only of the same, the work may be much shortened, by resolving each of the preceding equations into two. Thus,

$$\alpha+3\gamma+5\varepsilon+\&c.=G. \quad (=-2c-3c^3-\tfrac{15}{4}c^5-\&c.)$$

$$\alpha+3^3\gamma+5^3\varepsilon+\&c.=H. \quad (=-2c-29c^3-\tfrac{499}{4}c^5-\&c.)$$

$$\alpha+3^5\gamma+5^5\varepsilon+\&c.=I. \quad (=-2c-263c^3-\tfrac{13631}{4}c^5-\&c.)$$

$$(3^2-1)\alpha-5\,(5^2-3^2)\varepsilon=3^2G-H.$$

$$(3^2-1)\alpha-5^2(5^2-3^2)\varepsilon=3^2H-I.$$

$$(5^2-1)\,.\,(3^2-1)\alpha \;=5^2\,.\,3^2G-(5^2+3^2)\,H+I.$$

$$\alpha=\tfrac{225}{192}G-\tfrac{34}{192}H+\tfrac{1}{192}I.$$

$$\tfrac{225}{192}G=-\tfrac{225}{96}c-\tfrac{675}{192}c^3-\tfrac{3375}{768}c^5.$$

$$-\tfrac{34}{192}H=+\tfrac{34}{96}c+\tfrac{986}{192}c^3+\tfrac{16966}{768}c^5.$$

$$+\tfrac{1}{192}I=-\tfrac{1}{96}c-\tfrac{263}{192}c^3-\tfrac{13631}{768}c^5.$$

$$=\alpha \quad =-2c+\tfrac{1}{4}c^3-\tfrac{5}{96}c^5.$$

To compute, then, the coefficients α, ϵ, γ, &c. true to the ninth power of c, the process will be as follows:

First, compute the five series marked A, B, C, D, E, to c^9; and let the odd terms of each,

with their signs, be ..... G, H, I, K, L,

and the even terms of the same, ... N, O, P, Q, R,

so that $A = \overline{G + N}$,

$B = \overline{H + O}$,

$C = \overline{I + P}$, &c.

Next, compute the odd terms α, γ, ε, &c. from the following equations:

$$1 \cdot (9^2 - 1) \cdot (7^2 - 1) \cdot (5^2 - 1) \cdot (3^2 - 1) \cdot \alpha$$
$$= 9^2 \cdot 7^2 \cdot 5^2 \cdot 3^2 \cdot G - pH + qI - rK + L.$$

$$3 \cdot (9^2 - 3^2) \cdot (7^2 - 3^2) \cdot (5^2 - 3^2) \cdot \gamma$$
$$= 9^2 \cdot 7^2 \cdot 5^2 \cdot G' - qH' + rI' - K'.$$

$$5 \cdot (9^2 - 5^2) \cdot (7^2 - 5^2) \cdot \varepsilon = 9^2 \cdot 7^2 \cdot G'' - rH'' + I''.$$

$$7 \cdot (9^2 - 7^2) \cdot \eta = 9^2 \cdot G''' - H'''.$$

$$9 \cdot \iota = G''''.$$

Where p = sum of the products of every three of the squares in the preceding term,

q = sum of the products of every two,

r = sum of every one,

$G' = \overline{G - \alpha}$,

$G'' = \overline{G - \alpha - 3\gamma}$,

$G''' = \overline{G - \alpha - 3\gamma - 5\varepsilon}$, &c.

$H' = \overline{H - \alpha}$,

$H'' = \overline{H - \alpha - 3^2\gamma}$, &c. &c.

Lastly, compute the even terms ζ, δ, &c. from these equations:

$$2.(8^2-2^2).(6^2-2^2).(4^2-2^2).\zeta = 8^2.6^2.4^2.N - qO + rP - Q.$$

$$4.(8^2-4^2).(6^2-4^2).\delta = 8^2.6^2.N' - rO' + P'.$$

$$6.(8^2-4^2).\zeta = 8^2.N'' - O''.$$

$$8.\theta = N'''.$$

FINIS.

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part 2

ELEMENTS

OF

SPHERICAL TRIGONOMETRY,

AND OF

NAUTICAL AND PRACTICAL ASTRONOMY.

OF SINES, TANGENTS, SECANTS AND VERSED SINES.

LET a, b (fig. 1.) represent two arcs of the same circle, of which a is the greater, and s, d their sum and difference: Then, each of these four arcs, a, b, s, d, being supposed less than a quadrant, we have the following propositions:

1. Rad. = sin. 90° = cos. 0° = tan. 45° = $\frac{1}{2}$ sec. 60° = ver. 90° = cho. 60°.
2. Rad. = ver. a + cos. a = suvers. a — cos. a = ver. b + cos. b = &c.
3. R^2 = sin.^{2} a + cos.^{2} a = sec.^{2} a — tan.^{2} a = sin.^{2} b + cos.^{2} b = &c.

4. $R^2 = \text{tan.}\, a . \text{cot.}\, a = \text{sin.}\, a . \text{cosec.}\, a = \text{cos.}\, a . \text{sec.}\, a = \text{tan.}\, b . \text{cot.}\, b = \text{&c.}$

5. $R^2 : \text{tan.}^2 a :: \text{cot.}\, a : \text{tan.}\, a$, and $\text{tan.}\, a : \text{tan.}\, b :: \text{cot.}\, b : \text{cot.}\, a$, &c. [4.]

6. $R . \text{tan.}\, a = \text{sin.}\, a . \text{sec.}\, a = \text{sec.}\, a . \text{cosec.}\, a - R . \text{cot.}\, a.$

7. $R . \text{sec.}\, a = \text{tan.}\, a . \text{cosec.}\, a = \text{sin.}\, a . \text{tan.}\, a + R . \text{cos.}\, a.$

8. $R . \text{ver.}\, a = 2 \text{ sin.}^2 \tfrac{1}{2} a = \text{sin.}\, a . \text{tan.}\, \tfrac{1}{2} a = \tfrac{1}{2} \text{ chord}^2 a.$

9. $R . \text{suvers.}\, a = 2 \text{ cos.}^2 \tfrac{1}{2} a = \text{sin.}\, a . \text{cot.}\, \tfrac{1}{2} a.$

10. Suvers. a : ver. $a :: R + \text{cos.}\, a : R - \text{cos.}\, a :: R^2 : \text{tan.}^2 \tfrac{1}{2} a.$ [8, 9, 5, 2.]

11. $R + \text{cos.}\, a : R + \text{cos.}\, b :: R^2 + \text{tan.}^2 \tfrac{1}{2} b : R^2 + \text{tan.}^2 \tfrac{1}{2} a.$ [10.]

12. $R . \text{sin.}\, a = \text{cos.}\, a . \text{tan.}\, a = 2 \text{ sin.}\, \tfrac{1}{2} a . \text{cos.}\, \tfrac{1}{2} a.$

13. $R . \text{cos.}\, a = \text{sin.}\, a . \text{cot.}\, a =$ [2, 8.] $R^2 - 2 \text{sin.}^2 \tfrac{1}{2} a = \text{cos.}^2 \tfrac{1}{2} a - \text{sin.}^2 \tfrac{1}{2} a.$ [3.]

14. $R . (\text{sec.}\, a - R) = \text{tan.}\, a . \text{tan.}\, \tfrac{1}{2} a.$ [8.]

15. $R . (\text{sec.}\, a + R) = \text{tan.}\, a . \text{cot.}\, \tfrac{1}{2} a.$ [9.]

16. $2 \text{ cosec.}\, a = \text{cot.}\, \tfrac{1}{2} a + \text{tan.}\, \tfrac{1}{2} a.$ [14, 15.]

17. $2 \text{ cot.}\, a = \text{cot.}\, \tfrac{1}{2} a - \text{tan.}\, \tfrac{1}{2} a.$ [14, 15.]

18. $\text{Cosec.}\, a + \text{cot.}\, a = \text{cot.}\, \tfrac{1}{2} a$; or $\text{sec.}\, a + \text{tan.}\, a = \text{tan.}\, (45 + \tfrac{1}{2} a)$. [16, 17.]

19. $\text{Cosec.}\, a - \text{cot.}\, a = \text{tan.}\, \tfrac{1}{2} a$; or $\text{sec.}\, a - \text{tan.}\, a = \text{tan.}\, (45 - \tfrac{1}{2} a)$. [16, 17.]

20. $R^4 : R^4 \sim \tan.^2 a :: R . \sec. 2a : \sec.^2 a.$ [8, 9, 10, 13.]

These propositions are evident. The next may be thus demonstrated: In the circle AMN, (fig. 1.) whose centre is C, let the arc AB (=AG)=a, and AD (=Ad)=b, join BG, Dd, Bd, and dG, and let fall the perpendiculars dF, CH, upon BG, dG respectively. Because CH is perpendicular to the chord dG, CH produced must bisect the arc dG; hence the angle dCH at the centre, standing upon the half arc, is equal to the angle dBF at the circumference, standing upon the whole arc. Therefore the right angled triangles CHd, BFd are similar, and Cd . BF = Bd . CH, that is, $R . (\sin. a + \sin. b) = 2 \sin. \frac{1}{2} s . \cos. \frac{1}{2} d$, which is the 21st proposition: The rest easily follow from those that precede.

21. $R . (\sin. a + \sin. b) = 2 \sin. \frac{1}{2} s . \cos. \frac{1}{2} d.$

22. $R . (\sin. a - \sin. b) = 2 \cos. \frac{1}{2} s . \sin. \frac{1}{2} d.$ [21.]

23. $R . (\cos. a + \cos. b) = 2 \cos. \frac{1}{2} s . \cos. \frac{1}{2} d.$ [21.]

24. $R . (\cos. b - \cos. a) = 2 \sin. \frac{1}{2} s . \sin. \frac{1}{2} d.$ [22.]

Hence, because $a = \frac{1}{2}(s + d)$, and $b = \frac{1}{2}(s - d)$,

25. $R . (\sin. s + \sin. d) = 2 \sin. a . \cos. b.$ [21.]

26. $R . (\sin. s - \sin. d) = 2 \cos. a . \sin. b.$ [22.]

27. R . (cos.s + cos. d) = 2 cos. a . cos. b. [23.]

28. R . (cos.d—cos.s) = 2 sin. a . sin. b. [24.]

And, by addition and subtraction,

29. R. sin. s = sin. a . cos. b + cos. a . sin. b. [25, 26.]

30. R. sin. d = sin. a . cos. b — cos. a . sin. b. [25, 26.]

31. R. cos. s = cos. a . cos. b — sin. a . sin. b. [27, 28.]

32. R. cos. d = cos. a . cos. b + sin. a . sin. b. [27, 28.]

which, [by 12. 13.] may be also expressed thus,

33. R . sin. s = (tan. a + tan. b) . cos. a . cos. b
= (cot. a + cot. b) . sin. a . sin. b.

34. R . sin. d = (tan. a — tan. b) . cos. a . cos. b
= (cot. b — cot. a) . sin. a . sin. b.

35. R . cos. s = (cot. a — tan. b) . sin. a . cos. b. [31.]

36. R . cos. d = (cot. a + tan. b) . sin. a . cos. b. [32.]

From theorems 23, 24, we deduce the following:

37. Cos. s . cos. d = $\left(\frac{1}{2}\right.$ R . (cos. $2a$ + cos. $2b$) =$\left.\right)$
cos.^{2} a — sin.^{2} b. [23, 13.]

38. Sin. s . sin. d = sin.^{2} a — sin.^{2} b = cos.^{2} b — cos.^{2} a.
[24, 13.]

Which may be otherwise expressed as follows:

39. Cos. $\frac{1}{2}$ s . cos. $\frac{1}{2}$ d = cos.^{2} $\frac{1}{2}$ a — sin.^{2} $\frac{1}{2}$ b
= $\frac{1}{2}$ R . (suvers. a — ver. b). [37, 8, 9.]

40. Sin. $\frac{1}{2}s$. sin. $\frac{1}{2}d$ = sin.^{2} $\frac{1}{2}a$ — sin.^{2} $\frac{1}{2}b$ = $\frac{1}{2}$ R . (ver. a — ver. b). [38, 8, 9.]

41. Cos. a . cos. b = cos.^{2} $\frac{1}{2}s$ — sin.^{2} $\frac{1}{2}d$ = $\frac{1}{2}$ R . (suvers. s — ver. d). [37, 8, 9.]

42. Sin. a . sin. b = (sin.^{2} $\frac{1}{2}s$ — sin.^{2} $\frac{1}{2}d$) = $\frac{1}{2}$ R . (ver. s — ver. d). [38, 8, 9.]

From theorems 21—24 are deduced these analogies:

43. Sin. a + sin. b : sin. a — sin. b : : tan. $\frac{1}{2}s$: tan. $\frac{1}{2}d$. [21, 22, 12.]

44. Cos. a + cos. b : cos. b — cos. a : : cot. $\frac{1}{2}s$: tan. $\frac{1}{2}d$. [23, 24, 12.]

45. Cos. a + cos. b : sin. a + sin. b : : sin. a — sin. b : cos. b — cos. a : : R : tan. $\frac{1}{2}s$.

46. Cos. a + cos. b : sin. a — sin. b : : sin. a + sin. b : cos. b — cos. a : : R : tan. $\frac{1}{2}d$.

And, from theorems 33—36, these analogies:

47. Tan. a + tan. b : tan. a — tan. b : : sin. s : sin. d. [33, 34.]

48. Cot. a — tan. b : cot. a + tan. b : : cos. s : cos. d. [35, 36.]

49. R . (tan. a + tan. b) : R^2 + tan. a . tan. b : : sin. s : cos. d.

50. R . (tan. a — tan. b) : R^2 — tan. a . tan. b : : sin. d : cos. s. [34, 35, 12.]

Hence the tangents of the sum and difference may be expressed in terms of the tangents of the simple arcs.

51. Cot. s . (tan. a + tan. b = R^2 — tan. a . tan. b = tan. a . (cot. a — tan. b). [47, 50, 4.]

52. Cot. d . (tan. a — tan. b) = R^2 + tan. a . tan. b = tan. a . (cot. a + tan. b). [48, 50, 4.]

Whence, supposing, first $a = b$, next $a = 45°$, we have,

53. Cot. $2a$. 2 tan. a = R^2 — tan.^{2} a; or, 2 tan. a : tan. $2a$: : R^2 — tan.^{2} a : R^2.

54. Cot. $(45 - b)$. $(R$ — tan. $b) = R^2 + R$. tan. b; or, R — tan. b : R + tan. b : : R : tan. $(45 + b)$.

The following equations, which are only particular cases of theorems 21—24, are useful in the construction of tables: Suppose R = 1, S = sin. a, C = cos. a, and n any whole number.

55. Sin. na + sin. $(n+2).a = 2C.$ sin. $(n+1).a$. [21.]

56. Sin. $(n+2)\,a$ — sin. $na = 2S.$ cos. $(n+1).a$. [22.]

57. Cos. na + cos. $(n+2).a = 2C.$ cos. $(n+1).a$. [23.]

58. Cos. na — cos. $(n+2).a = 2S.$ sin. $(n+1).a$. [24.]

59. Sin. $(60 - a)$ + sin. a = sin. $(60 + a)$ [21.] when $(a + b) = 60$.

60. Sin. $(60-a)-\sin. a = \cos. (60+a).\sqrt{3}$. [22.]

61. $(\text{Cos. } a + \sin. a).\frac{1}{2}\sqrt{2} = \sin. (45 + a)$. [21.]; $(a + b) = 90$.

62. $(\text{Cos. } a - \sin. a).\frac{1}{2}\sqrt{2} = \cos. (45 + a)$. [22.]

63. $1 + \sin. b = 2 \sin.^2 (45 + \frac{1}{2}b)$. [21.]; $a = 90°$.

64. $1 - \sin. b = 2 \cos.^2 (45 + \frac{1}{2}b)$. [22.]

65. $1 + \cos. a = 2\cos.^2 \frac{1}{2}a = \sin. a . \cot. \frac{1}{2}a$. [23.] $b = 0$. [or 9.]

66. $1 - \cos. a = 2 \sin.^2 \frac{1}{2}a = \sin. a . \tan. \frac{1}{2}a$. [24, or 8.]

67. $\sqrt{(1+\sin. b)}-\sqrt{(1-\sin. b)}=2\sin.\frac{1}{2}b$. [63, 64, 22.]

68. $\sqrt{(1+\sin. b)}+\sqrt{(1-\sin. b)}=2\cos.\frac{1}{2}b$. [63, 64, 21.]

69. $\sqrt{(1+\cos. a)}-\sqrt{(1-\cos. a)}=2\cos. (45+\frac{1}{2}a)$. [65, 66, 22.]

70. $\sqrt{(1+\cos. a)}+\sqrt{(1-\cos. a)}=2\sin. (45+\frac{1}{2}a)$. [65, 66, 21.]

Let c be any arc less than a or b, then [29, 30.]

71. $\text{Sin. } b . \sin. (a + c)$
$= \sin. a . \sin. (b - c) + \sin. c . \sin. (a + b)$
$= \sin. a . \sin. (b + c) - \sin. c . \sin. (a - b)$
$= \cos. a . \cos. (b - c) - \cos. c . \cos. (a + b)$
$= \cos. c . \cos. (a - b) - \cos. a . \cos. (b + c)$.

72. Cos. b . sin. $(a + c)$

$= \cos. c \cdot \sin. (a + b) - \cos. a \cdot \sin. (b - c)$

$= \cos. c \cdot \sin. (a - b) + \cos. a \cdot \sin. (b + c)$

$= \sin. a \cdot \cos. (b - c) + \sin. c \cdot \cos. (a + b)$

$= \sin. a \cdot \cos. (b + c) + \sin. c \cdot \cos. (a - b)$.

73. And from these similar expressions may be obtained; for sin. b, or cos. b . cos. $(a + c)$, sin. b, or cos. b . sin. $(a - c)$, and sin. b, or cos. b . cos. $(a - c)$.

74. By theorem 26.

$2 \sin. (a + b) \cdot \sin. (b + c) = R \cdot \big(\cos. (a - c) - \cos. (a + 2b + c)\big)$,

$2 \sin. (a + b + c) \cdot \sin. b = R \cdot \cos. \big((a + c) - \cos. (a + 2b + c)\big)$,

and $2 \cos. (a + b + c) \cdot \cos. b = R \cdot \big(\cos. (a + c) + \cos. (a + 2b + c)\big)$.

Subtract the second equation from the first, and, to the latter part of the result, apply theorem 28: Also, add together the first and third equations, and, to the latter part of the sum, apply theorems 27——, and we have theorem 71, &c. &c.

OF THE SIGNS OF TRIGONOMETRICAL LINES.

75. The preceding theorems 2—74. remain true, whatever be the magnitude of the arcs a, b, c, d, s,

provided the signs be changed of the sine, cosine, &c. of every arc, whose magnitude is such as to render its sine, cosine, &c. negative; and the sign be changed of every right angle, whereof only one of its sides becomes negative.

The versed sines of all arcs are affirmative; but a being any arc less than a quadrant,

$$\left.\begin{matrix}\text{tan. sec.}\\ \text{sin. sec.}\\ \text{sin. tan.}\end{matrix}\right\} \text{ and consequently, } \begin{cases}\text{cot. cos. of } a + 90^\circ,\\ \text{cosec. cos. } a + 180^\circ,\\ \text{cosec. cot. } a + 270^\circ,\end{cases}$$

are negative.

The principle upon which this change of signs is founded, is simply this: Every trigonometrical line respecting an arc less than a quadrant being marked +, those lines respecting an arc greater than a quadrant, which are in an *opposite* direction to the former, must be designated.—Thus, the sine BE, of the arc AB, being *affirmative*, the sine GE, (of the arc ABMG,) in an opposite direction to BE, is *negative*. In like manner, AT, the tangent of the arc AB, (and also of the arc ABP,) being affirmative, AO, in an opposite direction to AT, the tangent of the arcs ABL, ABD, is negative.

If two arcs AB, BD be taken from the same point A in contrary directions, and the former considered

affirmative, the latter must be negative. Hence, the sine and tangent of a negative arc less than a quadrant are negative, but its cosine and versed sine are affirmative, (which may also be inferred from 8. 13.)

The following useful table, in which q denotes a quadrant, and z any smaller arc, exhibits the variation of the signs in respect to the sines and cosines of all arcs from 0 to 360°.

76. TABLE I.

1.	$\begin{cases}\sin.(+z) = +\sin.z.\\ \sin.(-z) = -\sin.z.\end{cases}$	1.	$\begin{cases}\cos.(+z) = +\cos.z.\\ \cos.(-z) = +\cos.z.\end{cases}$
2.	$\begin{cases}\sin.(q+z) = +\cos.z.\\ \sin.(q-z) = +\cos.z.\end{cases}$	2.	$\begin{cases}\cos.(q+z) = -\sin.z.\\ \cos.(q-z) = +\cos.z.\end{cases}$
3.	$\begin{cases}\sin.(2q+z) = -\sin.z.\\ \sin.(2q-z) = +\sin.z.\end{cases}$	3.	$\begin{cases}\cos.(2q+z) = -\cos.z.\\ \cos.2q-z = -\cos.z.\end{cases}$
4.	$\begin{cases}\sin.(3q+z) = -\cos.z.\\ \sin.(3q-z) = -\cos.z.\end{cases}$	4.	$\begin{cases}\cos.(3q+z) = +\sin.z.\\ \cos.(3q-z) = -\sin.z.\end{cases}$

This table need not be farther continued; because, if $4q$, or a whole circumference, be added to any arc on the left hand side of either part of the table, the sine or cosine on the right hand side, whether affirmative or negative, remains the same.

Before we proceed to such trigonometrical series' as are deducible from the preceding theory, it may be proper to shew how the numeral coefficients of the integral powers of a binomial may be obtained and expressed in general terms. They are to be found in the following table of numbers.

77. TABLE II.

1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	1,
2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,
3,	6,	10,	15,	21,	28,	36,	45,	55,	66,	78,
4,	10,	20,	35,	56,	84,	120,	165,	220,	286,	364,
5,	15,	35,	70,	126,	210,	330,	495,	715,	1001,	1365,
6,	21,	56,	126,	252,	462,	792,	1287,	2002,	3003,	4368,
7,	28,	84,	210,	462,	924,	1716,	3003,	5005,	8008,	12376,
8,	36,	120,	330,	792,	1716,	3432,	6435,	11440,	19448,	31824,
9,	45,	165,	495,	1287,	3003,	6435,	12870,	24310,	43758,	75582,
10,	55,	220,	715,	2002,	5005,	11440,	24310,	48620,	92378,	167960,
11,	66,	286,	1001,	3003,	8008,	19448,	43758,	92378,	184756,	352716,
12,	78,	364,	1365,	4368,	12376,	31824,	75582,	167960,	352716,	705432, &c.
&c.									&c.	

78. The table is constructed thus: In the first line each term is unit; in the second and every subsequent line, each term is the sum of those in the preceding line from the first, to the one under which it stands, inclusive.

So, 1, 1+4, 1+4+10, 1+4+10+20, &c. in the 4th line, make 1, 5, 15, 35, &c. which is the 5th.

Hence it follows, that the first line consists of the first differences of the numbers in the second, or of the second differences of those in the third, &c. so that the n^{th} differences of the numbers in the $(n + 1)$ line are each equal to 1.

Any line may be found independently of the preceding lines. To find the 5th, for example, put down 4 lines of numbers increasing by 1, beginning the first line with 1, the second with 2, the third with 3, &c. thus:

1,	2,	3,	4,	5,	6, &c.
2,	3,	4,	5,	6,	7, &c.
3,	4,	5,	6,	7,	8, &c.
4,	5,	6,	7,	8,	9, &c.
1,	5,	15,	35,	70,	126, &c.

Take the products of the corresponding terms, and divide each product by the first; the quotients are the numbers in the 5th line. Thus, $\frac{2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 5$, $\frac{3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 15$, $\frac{4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 35$, &c.

Of the numbers in the table, many curious properties may be observed; as,

1. The sum of the squares of the odd numbers in the second line continued to the number m, is equal to the number in the fourth line under m. Thus,

$1^2+3^2+5^2+7^2=84$, the num. in the 4th line under 7.

2. The same holds true of the even numbers. Thus,

$2^2+4^2+6^2+8^2=120$, the num. in the 4th line under 8.

3. The sum of the cubes of the numbers in the second line continued to the number m, is equal to the square of the number in the 3d line under m. Thus,

$1^3+2^3+3^3+4^3=10^2$, the num. in the 3d line under 4.

4. The sum of the numbers in any finite diagonal line is equal to that power of 2 which is denoted by the second number in the said diagonal line. Thus,

$1+5+10+10+5+1=2^5$, 5 being the 2d number.

5. The sums of the squares of the numbers in each finite diagonal line, taken in order from the beginning of the table, compose the numbers in the principal diagonal line, (1, 2, 6, 20, 70, &c.) Thus,

$$1^2+1^2=2,\ 1^2+2^2+1^2=6,\ 1^2+3^2+3^2+1^2=20,$$
$$1^2+4^2+6^2+4^2+1^2=70.$$

79. But, that these finite diagonal lines exhibit the numeral coefficients of the several affirmative integral

powers of a binomial, and the indefinite horizontal (or vertical) lines, those of the negative integral powers, is what we chiefly intended to observe.

AFFIRMATIVE INTEGRAL POWERS.	COEFFICIENTS.
$(1+x)^1 = 1+x.$	1, 1.
$(1+x)^2 = 1+2x+x^2.$	1, 2, 1.
$(1+x)^3 = 1+3x+3x^2+x^3.$	1, 3, 3, 1.
$(1+x)^4 = 1+4x+6x^2+4x^3+x^4.$	1, 4, 6, 4, 1.
&c.	&c.

NEGATIVE INTEGRAL POWERS.	COEFFICIENTS.
$(1+x)^{-1}=1-x+x^2-x^3+x^4-$&c.	1, 1, 1, 1, 1, &c.
$(1+x)^{-2}=1-2x+3x^2-4x^3+5x^4-$&c.	1, 2, 3, 4, 5, &c.
$(1+x)^{-3}=1-3x+6x^2-10x^3+15x^4-$&c.	1, 3, 6, 10, 15, &c.
$(1+x)^{-4}=1-4x+10x^2+20x^3-35x^4-$&c.	1, 4, 10, 20, 35, &c.
&c.	&c.

Hence, putting n for any number in the second vertical line, these, and other series in the table, may be expressed in general terms as follows:

In a diagonal line, from 1 to 1.

80. $1, n, \frac{n.n-1}{1.2}, \frac{n.n-1.n-2}{1.2.3}, \frac{n.n-1.n-2.n-3}{1.2.3.4}, \frac{n....(n-4)}{1.....5}$, &c.

In a horizontal or vertical line.

81. $1, n, \frac{n.n+1}{1.2}, \frac{n.n+1.n+2}{1.2.3}, \frac{n.n+1.n+2.n+3}{1.2.3.4}, \frac{n.....(n+4)}{1.....5}$, &c.

In a diagonal line, from 1 to ∞.

82. $1, n, \frac{n+1.n+2}{1.2}, \frac{n+2.n+3.n+4}{1.2.3}, \frac{n+3....n+6}{1...4}, \frac{n+4.....n+8}{1.....5}$, &c.

In a zig-zag diagonal line, from 1 to 1.

Such as, 1, 5 | 10, 20 | 15, 21 | 7, 8 | 1, 1.

83. $1, n, \frac{n-1.n}{1.2}, \frac{n-1.n.n+1}{1.2.3}, \frac{n-2....n+1}{1...4}, \frac{n-2....n+2}{1...5}, \frac{n-3...n+2}{1...6}$, &c.

In a zig-zag diagonal line, from 1 to ∞.

Such as, 1, 5 | 21, 56 | 210, 462 | 1716, 3432 | &c.

84. $1, n, \frac{n+1.n+2}{1.2}, \frac{n+1.n+2.n+3}{1.2.3}, \frac{n+2....n+5}{1...4}, \frac{n+2....n+6}{1...5}$ &c.

OF THE SINES, COSINES, &c. OF MULTIPLE ARCS.

By theorems 29. 31, the sines and cosines (and consequently tangents) of multiple arcs may be obtained: Thus, let sin. $a=s$, cos. $a=c$, tan. $a=t$, and radius$=1$, then,

85. TABLE III.

sin. $2a=2cs$.	cos. $2a=c^2-s^2$.
sin. $3a=3c^2s-s^3$.	cos. $3a=c^3-3cs^2$.
sin. $4a=4c^3s-4cs^3$.	cos. $4a=c^4-6c^2s^2+s^4$.
sin. $5a=5c^4s-10c^2s^3+s^5$.	cos. $5a=c^5-10c^3s^2+5cs^4$.
sin. $6a=6c^5s-20c^3s^3+6cs^5$.	cos. $6a=c^6-15c^4s^2+15c^2s^4-s^6$.
sin. $7a=7c^6s-35c^4s^3+21c^2s^5-s^7$.	cos. $7a=c^7-21c^5s^2+35c^3s^4-7cs^6$.
&c.	&c.

From this table it appears that the cosine of a multiple arc is composed of the odd terms, and the sine of the even terms of the corresponding power of the binomial $(c+s)$, with signs alternately affirmative and negative. Hence we have these general expressions:

86. Sin. $na = nc^{n-1}s - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} c^{n-3}s^3$

$+ \frac{n \ldots . n-4}{1 \ldots . 5} c^{n-5}s^5 - \frac{n \ldots . n-6}{1 \ldots 7} c^{n-7}s^7 +$ &c.

87. Cos. $na = c^n - \frac{n \cdot n-1}{1 \cdot 2} c^{n-2}s^2 + \frac{n \ldots n-3}{1 \ldots 4} c^{n-4}s^4$

$- \frac{n \ldots . n-5}{1 \ldots 6} c^{n-6}s^6 +$ &c.

Which, because $\frac{s}{c} = t$, may be changed to

88. Sin. $na = c^n \times \left(nt - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} t^3\right.$

$\left. + \frac{n \ldots n-4}{1 \ldots 5} t^5 - \frac{n \ldots n-6}{1 \ldots 7} t^7 + \text{&c.}\right)$

89. Cos. $na = c^n \times \left(1 - \frac{n \cdot n-1}{1 \cdot 2} t^2 + \frac{n \ldots n-3}{1 \ldots 4} t^4\right.$

$\left. - \frac{n \ldots . n-5}{1 \ldots 6} t^6 + \text{&c.}\right)$

And, dividing the former of these by the latter, we get,

90. Tan. na

$$= \frac{nt - \frac{n..n-2}{1..3} t^3 + \frac{n....n-4}{1...5} t^5 - \frac{n....n-6}{1...7} t^7 + \text{&c.}}{1 - \frac{n.n-1}{1 \cdot 2} t^2 + \frac{n...n-3}{1...4} t^4 - \frac{n....n-5}{1...6} t^6 + \text{&c.}}$$

Or,

$$\text{Tan. } na = nt + \frac{n.n^2-1}{1\ .\ 3}\,t^3 + \frac{n\ .\ n^2-1\ .\ 2n^2-3}{1\ .\ 3\ .\ 5}\,t^5 + \frac{n\ .\ n^2-1\ .\ (17n^4-53n^2+45)}{1\ .\ 3\ .\ 5\ .\ 7}\,t^7 + \&c.$$

Again, from the preceding Table the two following are easily deduced, by substituting $1-c^2$ for s^2, or $1-s^2$ for c^2.

91. TABLE IV.

$$\text{Sin. } 2a = 2cs.$$
$$\text{Sin. } 3a = 4c^2s - s.$$
$$\text{Sin. } 4a = 8c^3s - 4cs.$$
$$\text{Sin. } 5a = 16c^4s - 12c^2s + s.$$
$$\text{Sin. } 6a = 32c^5s - 32c^3s + 6cs.$$
$$\text{Sin. } 7a = 64c^6s - 80c^4s + 24c^2s - s.$$
$$\&c. \qquad\qquad \&c.$$

$$\text{Sin. } 3a = 3s - 4s^3.$$
$$\text{Sin. } 5a = 5s - 20s^3 + 16s^5.$$
$$\text{Sin. } 7a = 7s - 56s^3 + 112s^5 - 64s^7.$$
$$\text{Sin. } 9a = 9s - 120s^3 + 432s^5 - 576s^7 + 256s^9.$$

$$\text{Sin. } 2a = c\ .\ 2s.$$
$$\text{Sin. } 4a = c\ .\ (4s - 8s^3).$$
$$\text{Sin. } 6a = c\ .\ (6s - 32s^3 + 32s^5).$$
$$\&c. \qquad\qquad \&c.$$

92. TABLE V.

$$\text{Cos. } 2a = 2c^2 - 1.$$
$$\text{Cos. } 3a = 4c^3 - 3c.$$
$$\text{Cos. } 4a = 8c^4 - 8c^2 + 1.$$
$$\text{Cos. } 5a = 16c^5 - 20c^3 + 5c.$$
$$\text{Cos. } 6a = 32c^6 - 48c^4 + 18c^2 - 1.$$
$$\text{Cos. } 7a = 64c^7 - 112c^5 + 56c^3 - 7c.$$
&c. &c.

$$\text{Cos. } 2a = 1 - 2s^2.$$
$$\text{Cos. } 4a = 1 - 8s^2 + 8s^4.$$
$$\text{Cos. } 6a = 1 - 18s^2 + 48s^4 - 32s^6.$$

$$\text{Cos. } 3a = c.(1 - 4s^2).$$
$$\text{Cos. } 5a = c.(1 - 12s^2 + 16s^4).$$
$$\text{Cos. } 7a = c.(1 - 24s^2 + 80s^4 - 64s^6).$$
&c. &c.

From the sines of the odd multiples in the former table, and the cosines of the even multiples in the latter, we deduce these general theorems:

93. $$\text{Sin. } na = ns - \frac{n.n^2-1}{1.2.3}s^3 + \frac{n.n^2-1.n^2-9}{1.....5}s^5 - \frac{n.n^2-1.n^2-9.n^2-25}{1.....7}s^7 + \&c.$$

94. $\text{Cos. } na = 1 - \frac{n^2}{1.2}s^2 + \frac{n^2.n^2-4}{1....4}s^4$

$$-\frac{n^2.n^2-4.n^2-16}{1....6}s^6 + \frac{n^2.n^2-4.n^2-16.n^2-36}{1.....8}s^8 - \&c.$$

For, though the first series be obtained, by supposing n an odd number, and the second, by supposing n an even number; yet both series hold true, whether n be odd or even, integral or fractional. In the cases mentioned, they terminate; in others, they do not. Thus, if $n = 2$ in the former, we have,

$$\text{Sin. } 2a = 2s - s^2 - \tfrac{1}{4}s^5 - \tfrac{1}{8}s^7 - \tfrac{5}{32}s^9 - \&c.$$
$$= 2s\sqrt{(1-s^2)} = 2cs, \text{ as in Table III.}$$

In like manner, from the sines of the even multiples in the former, and the cosines of the odd multiples in the latter, we have these general theorems:

95. $\text{Sin. } na = c.\,(ns - \frac{n.n^2-4}{1.2.3}s^3 + \frac{n^2.n^2-4.n^2-16}{1...5}s^5$

$$-\frac{n.n^2-4.n^2-16.n^2-36}{1.....7}s^7 + \&c.)$$

96. $\text{Cos. } na = c.\,(1 - \frac{n^2-1}{1.2}s^2 + \frac{n^2-1.n^2-9}{1...4}s^4$

$$-\frac{n^2-1.n^2-9.n^2-25}{1.....6}s^6 + \&c.)$$

And, in either case, dividing the one series by the

other, we get this general expression for the tangent of a multiple arc:

$$97.\ \text{Tan. } na = ns + \frac{n.2n^2+1}{1.2.3}s^3 + \frac{n.(16n^4+20n^2+9)}{1\ .\ .\ .\ 5}s^5 + \frac{n.(272n^6+560n^4+518n^2+225)}{1\ .\ .\ .\ .\ 7}s^7 + \&c.$$

By putting $n = 1$, we get the tangent of an arc in terms of its sine, which may serve to prove the division, as it is otherwise known that

$$\text{Tan. } a = \frac{s}{\sqrt{(1-s^2)}} = s + \frac{1}{2}s^3 + \frac{1.3}{2.4}s^5 + \frac{1.3.5}{2.4.6}s^7 + \frac{1.3.5.7}{2.4.6.8}s^9 + \&c.$$

From Table V. we have also this theorem for the cosine of a multiple arc, n being any integral number, and $C = 2c$.

$$98.\ 2\,\text{Cos. } na = C^n - n\,.\,C^{n-2} + n\,.\,\frac{n-3}{2}\,C^{n-4} - n.\frac{n-4.n-5}{2\ .\ 3}C^{n-6} + n.\frac{n-5.n-6.n-7}{2\ .\ 3\ .\ 4}C^{n-8} - \&c.$$

From the same Table, the following curious and useful formulæ may be deduced:

Suppose $2\cos. a = x + \frac{1}{x}$;

then $2\cos. 2a = \left(x + \frac{1}{x}\right)^2 - 2 \qquad = x^2 + \frac{1}{x^2}.$

$2\cos. 3a = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) \qquad = x^3 + \frac{1}{x^3}.$

$2\cos. 4a = \left(x + \frac{1}{x}\right)^4 - 4\left(x + \frac{1}{x}\right)^2 + 2 = x^4 + \frac{1}{x^4}.$

&c. &c.

$2\cos. na = x^n + \frac{1}{x^n}.$

By solving the first quadratic equation, we find

$$x = \cos. a + \sin. a\sqrt{-1}.$$

$$\frac{1}{x} = \cos. a - \sin. a\sqrt{-1}.$$

By the last, $x^n = \cos. na + \sin. na\sqrt{-1}.$

$$\frac{1}{x^n} = \cos. na - \sin. na\sqrt{-1}.$$

Consequently,

$(\cos. a + \sin. a\sqrt{-1})^n = \cos. na + \sin. na\sqrt{-1}.$

$(\cos. a - \sin. a\sqrt{-1})^n = \cos. na - \sin. na\sqrt{-1}.$

POWERS OF THE SINE EXPRESSED BY THE SINES OF MULTIPLE ARCS.

THE following Tables, exhibiting the several powers of the sine and cosine of an arc, in terms of the sines

or cosines of its multiples, are easily derived from the preceding.

99. TABLE VI.

$$2s^2 = 1 - \cos. 2a.$$
$$4s^3 = 3 \sin. a - \sin. 3a.$$
$$8s^4 = 3 - 4 \cos. 2a + \cos. 4a.$$
$$16s^5 = 10 \sin. a - 5 \sin. 3a + \sin. 5a.$$
$$32s^6 = 10 - 15 \cos. 2a + 6 \cos. 4a - \cos. 6a.$$
$$64s^7 = 35 \sin. a - 21 \sin. 3a + 7 \sin. 5a - \sin. 7a.$$
$$128s^8 = 35 - 56 \cos. 2a + 28 \cos. 4a - 8 \cos. 6a + \cos. 8a.$$
$$256s^9 = 126 \sin. a - 84 \sin. 3a + 36 \sin. 5a - 9 \sin. 7a + \sin. 9a.$$

&c. &c.

100. TABLE VII.

$$2c^2 = 1 + \cos. 2a.$$
$$4c^3 = 3 \cos. a + \cos. 3a.$$
$$8c^4 = 3 + 4 \cos. 2a + \cos. 4a.$$
$$16c^5 = 10 \cos. a + 5 \cos. 3a + \cos. 5a.$$
$$32c^6 = 10 + 15 \cos. 2a + 6 \cos. 4a + \cos. 6a.$$
$$64c^7 = 35 \cos. a + 21 \cos. 3a + 7 \cos. 5a + \cos. 7a.$$

&c. &c.

In these Tables, the coefficients of the sines and cosines of the multiple arcs are the same for the corresponding powers of the sine and cosine of the single arc: But the signs of the terms in the latter Table are all affirmative; whereas, in the former, they are alternately affirmative and negative. It may suffice, therefore, to give the general expressions for the powers of the sine, which are as follows:

$$101.\ 2^{n-1}s^n = \frac{n.n-1.n-2\ldots\frac{n+3}{2}}{1.2.3\ldots\frac{n-1}{2}}\sin.a - \frac{n.n-1.n-2\ldots.\frac{n+5}{2}}{1.2.3\ldots\frac{n-3}{2}}\sin.3$$

$$+\frac{n.n-1.n-2..\frac{n+7}{2}}{1.2.3\ldots\frac{n-5}{2}}\sin.3a - \frac{n.n-1.n-2..\frac{n+9}{2}}{1.2.3\ldots\frac{n-7}{2}}\sin.7a$$

$$\ldots\ldots \pm \sin.na\ ;\ \text{when } n \text{ is odd.}$$

And when n is even,

$$102.\ 2^{n-1}s^n = \frac{1}{2}\cdot\frac{n.n-1.n-2\ldots\frac{n+2}{2}}{1.2.3\ldots\frac{n}{2}} - \frac{n.n-1.n-2\ldots\frac{n+4}{2}}{1.2.3\ldots\frac{n-2}{2}}\cos.2$$

$$+\frac{n.n-1.n-2\ldots\frac{n+6}{2}}{1.2.3\ldots\frac{n-4}{2}}\cos.4a - \frac{n.n-1.n-2\ldots\frac{n+8}{2}}{1.2.3\ldots\frac{n-6}{2}}\cos.6a$$

$$\ldots\ldots \pm \cos.na.$$

Or, inverting the order of the terms,

103. $\pm 2^{n-1} s^n = \sin. na - n \cdot \sin. (n-2) \cdot a$

$$+ \frac{n.n-1}{1 \cdot 2} \sin. (n-4).a - \frac{n.n-1.n-2}{1 \cdot 2 \cdot 3} \sin. (n-6).a$$

$$+ \frac{n....n-3}{1.... 4} \sin. (n-8).a ..... \pm \frac{n.n-1.n-2...\frac{n+3}{2}}{1 \cdot 2 \cdot 3 ...\frac{n-1}{2}} \sin. a.$$

$+$ if $n = 1, 5, 9, 13, 17,$ &c.
$-$ if $n = 3, 7, 11, 15, 19,$ &c.

104. $\pm 2^{n-1} s^n = \cos. na - n \cdot \cos. (n-2) \cdot a$

$$+ \frac{n.n-1}{1 \cdot 2} \cos. (n-4).a - \frac{n.n-1.n-2}{1 \cdot 2 \cdot 3} \cos. (n-6). a$$

$$+ \frac{n \cdot \cdot \cdot \cdot n-3}{1 \cdot \cdot \cdot \cdot 4} \cos. (n-8) \cdot a \cdot \cdot \cdot \cdot \cdot$$

$$\pm \frac{1}{2} \cdot \frac{n \cdot n-1 \cdot n-2 ... \frac{n+2}{2}}{1 \cdot 2 \cdot 3 .... \frac{n}{2}} \cdot$$

$+$ if $n = 4, 8, 12, 16,$ &c.
$-$ if $n = 2, 6, 10, 14,$ &c.

As an example of the utility of these Tables, let it be required to develope $(1 - 2s \cos. z)^n$ into a series of this form, viz.

$A + B \cdot \cos. z + C \cdot \cos. 2z + D \cdot \cos. 3z + E \cdot \cos. 4z +$ &c.

We have first,

$$(1 - 2\varepsilon \cos. z)^n = a + 2b \cos. z + 4c \cos.^2 z + 8d \cos.^3 z + 16e \cos.^4 z + \&c.;$$

where a, b, c, &c. denote the known terms of the nth power of the binomial $(1 - \varepsilon)$. Hence, by Table VII. we find,

105. $(1 - 2\varepsilon \cos. z)^n$

$= a + 2b.\cos. z + 2c.\cos. 2z + 2d.\cos. 3z + 2e.\cos. 4z + \&c.$

$+ 2c + 6d.\cos. z + 8e.\cos. 2z + 10f.\cos. 3z + \&c.$

$+ 6e + 20f.\cos. z + 30g.\cos. 2z + \&c.$

$+ 20g + 70h.\cos. z + \&c.$

&c. &c.

Thus,

$A = a + 2c + 6e + 20g + 70i + 252l + \&c.$

$B = 2b + 6d + 20f + 70h + 252k + \&c.$

$C = 2c + 8e + 30g + 112i + 420l + \&c.$

$D = 2d + 10f + 42h + 168k + 660m + \&c.$

$E = 2e + 12g + 56i + 240l + 990n + \&c.$

&c. &c.

The coefficients in the values of A and B are found in the principal diagonal line of Table II. And in the values of B, C, D, E, &c. each vertical column of coefficients being the differences of the coefficients of the next column, the law of continuation is manifest.

Thus, if $n = -2$, since

$$(1-\varepsilon)^{-2} = 1 + 2\varepsilon + 3\varepsilon^2 + 4\varepsilon^3 + 5\varepsilon^4 + 6\varepsilon^5 + 7\varepsilon^6 + \&c.$$
$$= a + b + c + d + e + f + g + \&c.$$

we have,

106. $(1-2\varepsilon\cos. z)^{-2}$

$$= 1 + 3.2\varepsilon^2 + 5.6\varepsilon^4 + 7.20\varepsilon^6 + \&c.$$
$$+ (2.2\varepsilon + 4.6\varepsilon^3 + 6.20\varepsilon^5 + 8.70\varepsilon^7 + \&c.).\cos. z.$$
$$+ (3\varepsilon^2 + 5.8\varepsilon^4 + 7.30\varepsilon^6 + 9.112\varepsilon^8 + \&c.).\cos. 2z.$$
$$+ (2.4\varepsilon^3 + 6.10\varepsilon^5 + 8.42\varepsilon^7 + 10.168\varepsilon^9 + \&c.).\cos. 3z.$$
$$\&c. \qquad\qquad \&c.$$
$$+ 2\varepsilon^n\Big((n+1) + \frac{n+2\,.\,n+3}{1}\varepsilon^2 + \frac{n+3\ldots n+5}{1\ .\ 2}\varepsilon^4$$
$$+ \frac{n+4\ldots.n+7}{1\ .\ 2\ .\ 3} + \&c.\Big)\cos. nz.$$

And, dividing the general term by the first, we have for the quotient,

$$2\varepsilon^n\Big((n+1) + \frac{n-1\,.\,n}{1}\varepsilon^2 + \frac{n-1\,.\,n\,.\,n+1}{1\ .\ 2}\varepsilon^4$$
$$+ \frac{n-1.n.n+1.n+4}{1\ .\ 2\ .\ 3}\varepsilon^6 + \frac{n-1.n.n+1.n+5.n+6}{1\ .\ 2\ .\ 3\ .\ 4}\varepsilon^8$$
$$+ \frac{n-1\,.\,n\,.\,n+1\,.\,n+6\,.\,n+7\,.\,n+8}{1\ .\ .\ .\ .\ 5}\varepsilon^{10} + \&c.\Big).$$

Now the first term,

$1 + 3.2\varepsilon^2 + 5.6\varepsilon^4 + 7.20\varepsilon^6 + \&c. = (1-4\varepsilon^2)^{-\frac{1}{2}} = s$ suppose; then the developement may be thus expressed:

107. $(1-2\varepsilon \cos. z)^{-s} = s+s.2\varepsilon^n\Big((n+1)+\frac{n-1\,.\,n}{1}\varepsilon^2$

$+\frac{n-1\,.\,n\,.\,n+1}{1\quad.\quad 2}\varepsilon^4+\&c.\Big)\cos. nz\,;$

where n is to be put $= 1, 2, 3,$ &c. successively.

SERIES OF SINES AND COSINES OF MULTIPLE ARCS.

From Table VII. and the first part of Table IV. we deduce the following

108. TABLE VIII.

Sin. $2a = 2s\,.\,\cos. a.$
Sin. $3a = 2s\,.\,(\frac{1}{2} + \cos. 2a).$
Sin. $4a = 2s\,.\,(\cos. a + \cos. 3a).$
Sin. $5a = 2s\,.\,(\frac{1}{2} + \cos. 2a + \cos. 4a).$
Sin. $6a = 2s\,.\,(\cos. a + \cos. 3a + \cos. 5a).$
Sin. $7a = 2s\,.\,(\frac{1}{2} + \cos. 2a + \cos. 4a + \cos. 6a).$
Sin. $8a = 2s\,.\,(\cos. a + \cos. 3a + \cos. 5a + \cos. 7a).$
&c. &c.

Putting n for any odd number, and m for any even number, we have these general terms:

109. Sin. $na = 2s \cdot \left(\frac{1}{2} + \cos. 2a + \cos. a + \cos. 6a + \ldots\ldots + \cos. (n-1).a\right)$.

110. Sin. $ma = 2s \cdot \left(\cos. a + \cos. 3a + \cos. 5a + \ldots\ldots + \cos. (m-1).a\right)$.

Now, supposing n and m indefinitely great, these series become equal; therefore,

111. $\frac{1}{2} = \cos. a - \cos. 2a + \cos. 3a - \cos. 4a + \cos. 5a - \cos. 6a +$ &c.

But,

112. $-\frac{1}{2} = \cos. a + \cos. 2a + \cos. 3a + \cos. 4a + \cos. 5a + \cos. 6a +$ &c.

(For, putting this last series $= S$, we have,

$$2S.\cos. a = 2\cos.^2 a + 2\cos. a.\cos. 2a + 2\cos. a.\cos. 3a + 2\cos. a . \cos. 4a + \text{&c.}$$

$$= 1 + \cos. 2a + \cos. 3a + \cos. 4a + \cos. 5a + \cos. 6a + \cos. a + \cos. 2a + \cos. 3a + \cos. 4a + \cos. 5a + \text{&c.}$$

$= 1 + 2S - \cos. a$, therefore $2S = \frac{1-\cos. a}{\cos. a - 1} = -1$).

Therefore, by adding [111, 112,]

113. * $0 = \cos. a + \cos. 3a + \cos. 5a + \cos. 7a + \cos. 9a +$ &c.

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* From these three equations (108, 109, 110,) it is easy to deduce, by the Theory of Analytical Functions, the sums of the follow-

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Again, suppose $na = 360°$, then sin. $na = 0$, and sin. $ma = \sin.(n+1).a = s$; therefore,

113. 1 $\left\{ -\frac{1}{2} = \cos. 2a + \cos. 4a + \cos. 6a + \cos. 8a + \ldots + \cos.(n-1).a. \right.$

113. 2 $\left\{ \frac{1}{2} = \cos. a + \cos. 3a + \cos. 5a + \ldots\ldots\ldots + \cos. na. \right.$

114. $0 = \cos. a + \cos. 2a + \cos. 3a + \cos. 4a \ldots + \cos. 360°.$

Again, if p be any odd part of the circumference, that is, if, for example, $7p = 360°$, then (113. 1),

$$\cos. 2p + \cos. 4p + \cos. 6p = -\tfrac{1}{2}.$$

But $\cos. 4p = +\cos. 3p$, and $\cos. 6p = +\cos. p$, therefore $\cos. 3p + \cos. 2p + \cos. p + \frac{1}{2} = 0$.

Now, let a be any arc less than p, and assume this series of equidifferent arcs, viz. :

$a, a+p, a+2p, a+3p, a+4p, a+5p, a+6p,$

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ing indefinite series of sines, viz. by multiplying each side of the equation by the prime function of a, and then finding their primitives,

1. $\frac{1}{2}a = \sin.a - \frac{1}{2}\sin.2a + \frac{1}{3}\sin.3a - \frac{1}{4}\sin.4a + \frac{1}{5}\sin.5a +$ &c.

2. $q - \frac{1}{2}a = \sin.a + \frac{1}{2}\sin.2a + \frac{1}{3}\sin.3a + \frac{1}{4}\sin.4a + \frac{1}{5}\sin.5a +$ &c.

3. $\frac{1}{2}q = \sin.a + \frac{1}{3}\sin.3a + \frac{1}{5}\sin.5a + \frac{1}{7}\sin.7a +$ &c.

where the constant quantity q (as appears from comparing the last of these with 152) is a quadrantal arc of the same circle.

the sums of the sines of the extreme terms, and of every two equally distant from them, (by 21.) give the following equations:

$$\sin. a + \sin. (a + 6p) = 2 \sin. (a + 3p).\cos. 3p.$$

$$\big(\sin. (a + p) + \sin. (a + 5p)\big) = 2 \sin. (a + 3p).\cos. 2p.$$

$$\sin. (a + 2p) + \sin. (a + 4p) = 2 \sin. (a + 3p).\cos. p,$$

to which adding $\sin. (a + 3p) = 2 \sin. (a + 3p).\ \frac{1}{2}$,

we have,

$$\sin. a + \sin. (a + p) + \sin. (a + 2p) + \sin. (a + 3p) + \sin. (a + 4p) + \sin. (a + 5p) + \sin. (a + 6p) = 0.$$

In the same manner, if n be any odd number whatever, and $np = 360°$, we find,

115. * $$\text{Sin. } a + \sin. (a + p) + \sin. (a + 2p) + \sin. (a + 3p) + \ldots\ldots + \sin. \big(a + (n - 1) . p\big) = 0.$$

We have taken a less than p, in order to get the most simple expression of the series at once. For, if $(a + p)$, $(a + 2p)$, &c. were substituted for a in the above series, it might easily be reduced to the same form.

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* Hence this geometrical theorem:

If the circumference of a circle be divided into any odd number of equal parts, and from any point in the circumference, straight lines be drawn to the several points of division, the sums of the alternate lines shall be equal.

The same things being supposed, it may be shewn in like manner, that,

116. $\text{Cos. } a + \text{cos. } (a + p) + \text{cos. } (a + 2p) + \text{cos.}(a + 3p) + \ldots\ldots + \text{cos.}\big(a + (n-1) \cdot p\big) = 0.$

From the same table, it is easy to deduce the following theorems, putting r for any integral number, m and n as before.

117. $\text{Cos.}a + \text{cos.}3a + \text{cos.}5a + \text{cos.}7a + \ldots\ldots + \text{cos.}na$

$$= \frac{\text{sin.}(n+1).a}{2\text{ sin. }a} \cdot = \frac{\text{sin. } \frac{n+1}{2} a \cdot \text{ cos. } \frac{n+1}{2} a}{\text{sin. } a}.$$

118. $\text{Cos.}2a + \text{cos.}4a + \text{cos.}6a + \text{cos.}8a + \ldots\ldots + \text{cos.}ma$

$$= \frac{\text{sin. } (m+1) \cdot a}{2\text{ sin. } a} - \tfrac{1}{2} = \frac{\text{cos. } \frac{m+2}{2} a . \text{sin. } \frac{1}{2}ma}{\text{sin. } a}.$$

119. $\text{Cos.}a + \text{cos.}2a + \text{cos.}3a + \text{cos.}4a + \ldots\ldots + \text{cos.}ra$

$$= \frac{\text{sin. } (r+1).a + \text{sin. } ra}{2\text{ sin. } a} - \tfrac{1}{2} = \frac{\text{cos. } \frac{r+1}{2} a . \text{sin. } \frac{1}{2}ra}{\text{sin. } \frac{1}{2}a}.$$

120. $\text{Cos.}a - \text{cos.}2a + \text{cos.}3a - \text{cos.}4a + \ldots\ldots + \text{cos.}na$

$$= \frac{\text{sin.}(n+1).a - \text{sin.}na}{2\text{ sin. } a} + \tfrac{1}{2} = \frac{\text{cos.} \frac{n+1}{2} a . \text{sin.} \frac{1}{2}na}{\text{cos. } \frac{1}{2}a}.$$

Similar theorems respecting the sines may be obtained thus. Put

$S = \text{sin.}a + \text{sin.}2a + \text{sin.}3a + \text{sin.}4a + \ldots\ldots + \text{sin.}ra.$

Then,

$$2S \cdot \cos.a$$
$$= \sin.2a+\sin.3a+\sin.4a+\ldots\ldots+\sin.(r+1).a.$$
$$+\sin.a+\sin.2a+\sin.3a+\ldots\ldots+\sin.(r-1).a.$$
$$= 2S-\sin.a+\sin.(r+1).a-\sin.ra.$$
$$= 2S-2\sin.\tfrac{1}{2}a.\cos.\tfrac{1}{2}a+2\cos.(ra+\tfrac{1}{2}a).\sin.\tfrac{1}{2}a.$$

Therefore,

$$S=\frac{\sin.\frac{1}{2}a\left(\cos.\frac{1}{2}a-\cos.(ra+\frac{1}{2}a)\right)}{1-\cos.a}=\frac{\sin.\frac{r+1}{2}a.\sin.\frac{1}{2}ra}{\sin.\frac{1}{2}a}.$$

Hence,

121. $$\text{Sin.}a+\sin.2a+\sin.3a+\sin.4a+\ldots+\sin.ra$$
$$=\frac{\sin.\frac{r+1}{2}a\,.\,\sin.\frac{1}{2}ra}{\sin.\frac{1}{2}a}.$$

In like manner,

122. $$\text{Sin.}a-\sin.2a+\sin.3a-\sin.4a+\ldots+\sin.na$$
$$=\frac{\sin.\frac{n+1}{2}a\,.\cos.\frac{1}{2}na}{\cos.\frac{1}{2}a}.$$

Consequently [29, 12.]

123. $$\text{Sin.}a+\sin.3a+\sin.5a+\sin.7a+\ldots+\sin.na$$
$$=\frac{\sin.^2\frac{n+1}{2}a}{\sin.a}.$$

Hence, putting $ra=90°$, we get the sum of the sines

of the arcs of a quadrant increasing by the first arc, viz.:

124. $\text{Sin.}\,a + \text{sin.}\,2a + \text{sin.}\,3a + \ldots\ldots + \text{sin.}\,90^\circ = \frac{1}{2}\text{cot.}\,\frac{1}{2}a + \frac{1}{2}$.

* But the same series continued indefinitely.

125. $\text{Sin.}a + \text{sin.}2a + \text{sin.}3a + \text{\&c.} = \frac{\text{sin.}\,a}{2(1-\text{cos.}a)} = \frac{1}{2}\text{cot.}\,\frac{1}{2}a.$

We have also,

126. $\text{Sin.}a - \text{sin.}2a + \text{sin.}3a - \text{sin.}4a + \ldots = \frac{\text{sin.}\,a}{2(1+\text{cos.}a)} = \frac{1}{2}\text{tan.}\,\frac{1}{2}a.$

And consequently [16.]

127. † $\text{Sin.}a + \text{sin.}3a + \text{sin.}5a + \text{sin.}7a + \ldots\ldots = \frac{1}{2}\text{cosec.}a.$

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* This seeming paradox may be thus explained: Since the sines of the first and second quadrants have the sin. 90° common to both, the sines of all the arcs of the first semicircle will be the double of those of the first quadrant lessened by 1, and therefore $=$ cot. $\frac{1}{2}a$. The sum of those arcs which terminate in the second semicircle will be the same, only negative, that is, $=$ — cot. $\frac{1}{2}a$: And so on. Thus, $\text{sin.}\,a + \text{sin.}\,2a + \text{sin.}\,3a + \text{\&c.} = \text{cot.}\,\frac{1}{2}a - \text{cot.}\,\frac{1}{2}a + \text{cot.}\,\frac{1}{2}a - \text{\&c.}$ But it appears from [108.], that $\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - \text{\&c.}$ Therefore also $\frac{1}{2}\text{cot.}\,\frac{1}{2}a = \text{cot.}\,\frac{1}{2}a - \text{cot.}\,\frac{1}{2}a + \text{\&c.} = \text{sin.}\,a + \text{sin.}\,2a + \text{\&c.}$

† Similar theorems to those noticed in a preceding note (p. 29), may be obtained in the same manner from 122, 123, 124:

Lastly, taking d in place of a for the common difference of the series of arcs, and proceeding in the same manner [117—118,] we shall have these more general theorems:

128. $$\text{Sin.}a + \text{sin.}(a+d) + \text{sin.}(a+2d) + \text{sin.}(a+3d) + \ldots\ldots + \text{sin.}(a+rd) = \frac{\text{sin.}(a+\frac{1}{2}rd).\text{sin.}\frac{r+1}{2}d}{\text{sin.}\frac{1}{2}d}.$$

129. $$\text{Sin.}a - \text{sin.}(a+d) + \text{sin.}(a+2d) - \text{sin.}(a+3d) + \ldots\ldots + \text{sin.}(a+md) = \frac{\text{sin.}(a+\frac{1}{2}md).\text{cos.}\frac{m+1}{2}d}{\text{cos.}\frac{1}{2}d}.$$

130. $$\text{Cos.}a + \text{cos.}(a+d) + \text{cos.}(a+2d) + \text{cos.}(a+3d) + \ldots\ldots + \text{cos.}(a+rd) = \frac{\text{cos.}(a+\frac{1}{2}rd).\text{sin.}\frac{r+1}{2}d}{\text{sin.}\frac{1}{2}d}.$$

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1. $\text{log.}\frac{1}{2\,\text{sin.}\frac{1}{2}a} = \text{cos.}\,a + \frac{1}{2}\text{cos.}\,2a + \frac{1}{3}\text{cos.}\,3a + \frac{1}{4}\text{cos.}\,4a + \&c.$

2. $\text{log.}\,2\,\text{cos.}\frac{1}{2}a = \text{cos.}\,a - \frac{1}{2}\text{cos.}\,2a + \frac{1}{3}\text{cos.}\,3a - \frac{1}{4}\text{cos.}\,4a + \&c.$

3. $\frac{1}{2}\text{log.}\,\text{cot.}\frac{1}{2}a = \text{cos.}\,a + \frac{1}{3}\text{cos.}\,3a + \frac{1}{5}\text{cos.}\,5a + \frac{1}{7}\text{cos.}\,7a + \&c.$

Several theorems may also be deduced from these, without having recourse to any other principles than such as are here explained. Thus, from 123, 122, we find,

$$\frac{4\,\text{sin.}\,a}{5 + 4\,\text{cos.}\,a} = \text{sin.}\,a - \frac{1}{2}\text{sin.}\,2a + \frac{1}{4}\text{sin.}\,3a - \frac{1}{8}\text{sin.}\,4a + \&c.$$

$$\frac{4\,\text{sin.}\,a}{5 - 4\,\text{cos.}\,a} = \text{sin.}\,a + \frac{1}{2}\text{sin.}\,2a + \frac{1}{4}\text{sin.}\,3a + \frac{1}{8}\text{sin.}\,4a + \&c.$$

191. $\text{Cos.}a - \cos.(a+d) + \cos.(a+2d) - \cos.(a+3d)$
$+ \ldots\ldots + \cos.(a+nd) = \dfrac{\cos.(a+\frac{1}{2}nd)\sin.\frac{1}{2}(n+1)d}{\sin.\frac{1}{2}d}.$

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192. $\text{Sin.}a + \sin.(a+d) + \sin.(a+2d) + \&c. \ldots$
$= \dfrac{\cos.(a-\frac{1}{2}d)}{2\sin.\frac{1}{2}d}.$

193. $\text{Sin.}a - \sin.(a+d) + \sin.(a+2d) - \&c. \ldots$
$= \dfrac{\sin.(a-\frac{1}{2}d)}{2\cos.\frac{1}{2}d}.$

194. $\text{Cos.}a + \cos.(a+d) + \cos.(a+2d) + \&c. \ldots$
$= -\dfrac{\sin.(a-\frac{1}{2}d)}{2\sin.\frac{1}{2}d}.$

195. $\text{Cos.}a - \cos.(a+d) + \cos.(a+2d) - \&c. \ldots$
$= \dfrac{\cos.(a-\frac{1}{2}d)}{2\cos.\frac{1}{2}d}.$

The use of these series' may be exemplified, by shewing that the seventeenth part of the circumference of a circle can be determined geometrically.

Thus, let a denote the seventeenth part of the circumference to the radius 1, then,

$\sin.2a + \cos.4a + \cos.6a + \cos.8a + \cos.10a + \cos.12a$
$+ \cos.14a + \cos.16a = -\frac{1}{2}.$

Now, this series of cosines being denoted by S, let it be divided into these two parts, viz. :

$$\cos. 2a + \cos. 4a + \cos. 8a + \cos. 16a = x.$$
$$\cos. 6a + \cos. 10a + \cos. 12a + \cos. 14a = y.$$

And their product [by 27] will be found to be,

$$\tfrac{1}{2} \cdot 4S = xy, \text{ therefore } xy = -1.$$
$$\text{Thus, } x + y = -\tfrac{1}{2},$$
$$\text{and } xy = -1.$$

Hence both x and y may be determined geometrically.

Again, let x be divided as follows, viz. :

$$\cos. 2a + \cos. 8a = v.$$
$$\cos. 4a + \cos. 16a = z.$$

And their product will be found in the same manner to be $\frac{1}{2}S = vz$, therefore $vz = -\frac{1}{4}$.

Also, dividing y as follows, viz. :

$$\cos. 6a + \cos. 10a = v'.$$
$$\cos. 12a + \cos. 14a = z'.$$

The product of these will likewise be $\frac{1}{2}S = v'z'$, therefore $v'z' = -\frac{1}{4}$.

Hence, v, z, and v', z', may all be determined geometrically.

Lastly, because $\cos. 4a + \cos. 16a \ldots\ldots = z.$

and $\cos. 4a \cdot \cos. 16a \left(= \frac{1}{2}.(\cos. 12a + \cos. 14a) \right) = \frac{1}{2}z'.$

131. $\text{Cos.}a - \text{cos.}(a+d) + \text{cos.}(a+2d) - \text{cos.}(a+3d)$
$+ \ldots\ldots + \text{cos.}(a+md) = \dfrac{\text{cos.}(a+\frac{1}{2}md).\text{cos.}\frac{m+1}{2}d}{\text{cos.}\frac{1}{2}d}.$

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132. $\text{Sin.}a + \text{sin.}(a+d) + \text{sin.}(a+2d) + \&\text{c.} \ldots$
$= \dfrac{\text{cos.}(a-\frac{1}{2}d)}{2\,\text{sin.}\frac{1}{2}d}.$

133. $\text{Sin.}a - \text{sin.}(a+d) + \text{sin.}(a+2d) - \&\text{c.} \ldots$
$= \dfrac{\text{sin.}(a-\frac{1}{2}d)}{2\,\text{cos.}\frac{1}{2}d}.$

134. $\text{Cos.}a + \text{cos.}(a+d) + \text{cos.}(a+2d) + \&\text{c.} \ldots$
$= -\dfrac{\text{sin.}(a-\frac{1}{2}d)}{2\,\text{sin.}\frac{1}{2}d}.$

135. $\text{Cos.}a - \text{cos.}(a+d) + \text{cos.}(a+2d) - \&\text{c.} \ldots$
$= \dfrac{\text{cos.}(a-\frac{1}{2}d)}{2\,\text{cos.}\frac{1}{2}d}.$

The use of these series' may be exemplified, by shewing that the seventeenth part of the circumference of a circle can be determined geometrically.

Thus, let a denote the seventeenth part of the circumference to the radius 1, then,

$$\text{cos.}\,2a + \text{cos.}\,4a + \text{cos.}\,6a + \text{cos.}\,8a + \text{cos.}\,10a + \text{cos.}\,12a + \text{cos.}\,14a + \text{cos.}\,16a = -\tfrac{1}{2}.$$

Now, this series of cosines being denoted by S, let it be divided into these two parts, viz. :

$$\cos. 2a + \cos. 4a + \cos. 8a + \cos. 16a = x.$$
$$\cos. 6a + \cos. 10a + \cos. 12a + \cos. 14a = y.$$

And their product [by 27] will be found to be,

$$\tfrac{1}{2} . 4S = xy, \text{ therefore } xy = -1.$$
$$\text{Thus, } x + y = -\tfrac{1}{2},$$
$$\text{and } \quad xy = -1.$$

Hence both x and y may be determined geometrically.

Again, let x be divided as follows, viz. :

$$\cos. 2a + \cos. 8a = v.$$
$$\cos. 4a + \cos. 16a = z.$$

And their product will be found in the same manner to be $\frac{1}{2}S = vz$, therefore $vz = -\frac{1}{4}$.

Also, dividing y as follows, viz. :

$$\cos. 6a + \cos. 10a = v'.$$
$$\cos. 12a + \cos. 14a = z'.$$

The product of these will likewise be $\frac{1}{2}S = v'z'$, therefore $v'z' = -\frac{1}{4}$.

Hence, v, z, and v', z', may all be determined geometrically.

Lastly, because $\cos. 4a + \cos. 16a \ldots\ldots = z.$

and $\cos. 4a . \cos. 16a \left(= \frac{1}{2}.(\cos. 12a + \cos. 14a) \right) = \frac{1}{2}z'.$

Cos. $16a$, or (which is the same) cos. a may likewise be determined.

Hence this simple geometrical construction:

Let C (fig. 2.) be the centre of the circle, and ACD a diameter. Make AB perpendicular to AC, and equal to $\frac{1}{2}$AC. Draw DBF, on which take BE, BF, each equal to BA. Make EH, FG each perpendicular to DB, and equal to AC. Bisect DE in I, and make IK = EB. Join IH, KG, on DB, take IM = IH, and on KG, KO = KF. Divide DM in N, so that the rectangle DNM = FG, GO. [To do this, erect the perpendiculars MT and DS equal to AC and GO, join SF, and upon it describe a semicircle cutting DB in N.] Apply the chord DP = DN, and draw the radius CR parallel to DP; then is AR the arc required.

THE SINE, TANGENT, &c. EXPRESSED BY INFINITE SERIES, IN TERMS OF THE ARC.

To express the sine, tangent, &c. of an arc, by infinite series, in terms of that arc, let x denote an indefinitely small arc, whose sine may therefore be expressed

by the same quantity x, and its cosine by the radius ($= 1$); let A denote any arc of a given magnitude, whose sin. $= s$, cos. $= c$, and tan. $= t$. And [by 29, 31] let a table of the sines and cosines of the arcs $(A + x)$, $(A + 2x)$, $(A + 3x)$, &c. be formed. Thus,

ARCS.	SINES.	COSINES.
$A+x$	$\ldots s+cx.$	$c-sx.$
$A+2x$	$\ldots s+2cx-sx^2.$	$c-2sx-cx^2.$
$A+3x$	$\ldots s+3cx-3sx^2-cx^3.$	$c-3sx-3cx^2+sx^3.$
$A+4x$	$\ldots s+4cx-6sx^2-4cx^3+sx^4.$	$c-4sx-6cx^2+4sx^3+cx^4.$
&c.	&c.	&c.

In this table we observe that the affirmative and negative signs in each term follow one another alternately, in pairs (if we except the first term of the cosine); and that the powers and numeral coefficients of x are the same with those in the corresponding power of the binomial $(1 + x)$. Taking n therefore to denote any integral number, the sine and cosine of $(A + nx)$ may be thus expressed:

136. $$\text{Sin.}(A+nx)=s+nxc-\frac{n.n-1}{1\,.\,2}x^2s-\frac{n.n-1.n-2}{1\,.\,2\,.\,3}x^3c + \frac{n\ldots n-3}{1\ldots 4}x^4s + \&c.$$

137. $\text{Cos.}(A+nx)=c-nxs-\frac{n.n-1}{1\,.\,2}x^2c+\frac{n.n-1.n-2}{1\,.\,2\,.\,3}x^3s$

$$+\frac{n\,\ldots\,n-3}{1\,\ldots\,4}x^4c-\text{\&c.}$$

Now, supposing $nx=a$, an arc of a given magnitude, it is plain that in whatever ratio we conceive x to be diminished, n must be augmented in the reciprocal ratio; but x, on the preceding hypothesis, is indefinitely small, therefore n must be indefinitely great; so that n, $n-1$, $n-2$, &c. may, for any given number of terms, be reckoned equal. Hence,

138. $\text{Sin.}(A+a)=s+ac-\frac{1}{2}a^2s-\frac{1}{2.3}a^3c+\frac{1}{2.3.4}a^4s$

$$+\frac{1}{2.3.4.5}a^5c-\text{\&c.}$$

139. $\text{Cos.}(A+a)=c-as-\frac{1}{2}a^2c+\frac{1}{2.3}a^3s+\frac{1}{2.3.4}a^4c$

$$-\frac{1}{2.3.4.5}a^5s-\text{\&c.}$$

140. $\text{Tan.}(A+a)=t+\frac{a}{c^2}+\frac{a^2t}{c^2}+\frac{3-2c^2}{3c^4}.a^3$

$$+\frac{3t-3c^2t}{3c^4}.a^4+\text{\&c.}$$

If $A=45^\circ$, then $s=c=\frac{1}{2}\sqrt{2}$, and,

141. $\text{Sin.}(45+a)=s(1+a-\frac{1}{2}a^2-\frac{1}{2.3}a^3+\frac{1}{2.3.4}a^4$

$$+\frac{1}{2\,\ldots\,5}a^5-\text{\&c.})$$

142. $\text{Sin.}(45-a)=s(1-a-\frac{1}{2}a^2+\frac{1}{2.3}a^3+\frac{1}{2.3.4}a^4$
$-\frac{1}{2\ldots5}a^5-\text{\&c.})$

Dividing one of these series by the other, we get,

143. $\text{Tan.}(45+a)=1+2a+2a^2+\frac{8}{3}a^3+\frac{10}{3}a^4+\frac{64}{15}a^5$
$+\frac{244}{45}a^6+\frac{2176}{315}a^7+\text{\&c.}$

144. $\text{Tan.}(45-a)=1-2a+2a^2-\frac{8}{3}a^3+\frac{10}{3}a^4-\frac{64}{15}a^5$
$+\frac{244}{45}a^6-\text{\&c.}$

Also, taking their sum and difference, and dividing by $2s$, we find [25, 26.]

145. $\text{Sin. } a=a-\frac{1}{1.2.3}a^3+\frac{1}{1\ldots5}a^5-\frac{1}{1\ldots7}a^7$
$+\frac{1}{1\ldots9}a^9-\text{\&c.}$

146. $\text{Cos. } a=1-\frac{1}{1.2}a^2+\frac{1}{1\ldots4}a^4-\frac{1}{1\ldots6}a^6$
$+\frac{1}{1\ldots8}a^8-\text{\&c.}$

Again, substitute $\frac{1}{2}a$ for a, [in 143, 144,] and taking half the sum and half the difference, [18, 19.]

147. * $\text{Tan.}a = a + \frac{1}{3}a^3 + \frac{2}{15}a^5 + \frac{17}{315}a^7 + \frac{62}{2835}a^9 + \&c.$

148. $\text{Sec.}a = 1 + \frac{1}{2}a^2 + \frac{5}{24}a^4 + \frac{61}{720}a^6 + \frac{277}{8064}a^8 + \&c.$

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* The series 147, 148 may also be found from 145, 146, (because $\text{tan.} = \frac{\text{sin.}}{\text{cos.}}$ and $\text{sec.} = \frac{1}{\text{cos.}}$). From an attentive view of the operations, the following rules for finding the coefficients may be deduced:

1. In the series for the tangent, let n denote the index of a in any term, c the coefficient of the term whose index is $\frac{n-1}{2}$ (if there be any such term), p the product of the coefficients of two terms, the sum of whose indices is $(n-1)$, p' the product of the coefficients of other two terms, the sum of whose indices is the same, &c.; then,

$\frac{1}{n} \cdot (c^2 + 2p + 2p' + 2p'' + \&c.)$ is the coefficient of the term, whose index is n.

Thus, if $n=11$, $\frac{1}{n}(c^2+2p+2p') = \frac{1}{11}\left(\frac{4}{225} + \frac{34}{945} + \frac{124}{2835}\right) = \frac{1382}{155975}$,

if $n = 13$, $\frac{1}{n} \cdot (2p + 2p' + 2p'') = \frac{21844}{6081075}$, here $c = 0$.

2. In the series for the secant, let n, p, p', &c. be as before.

Then $\frac{1}{n}(p + p' + p'' + \&c.)$ is the coefficient of the term, whose index is n.

Also, because $\frac{\text{cos.}}{\text{sin.}} = \text{cot.}$

149. $\text{Cot. } a = \frac{1}{a} - \frac{1}{3}a - \frac{1}{45}a^3 - \frac{2}{945}a^5 - \frac{1}{4725}a^7 - \frac{2}{93555}a^9 - \text{\&c.}$

And, because ver. = rad. — cos.

150. $\text{Ver. } a = \frac{1}{2}a^2 - \frac{1}{1.2.3.4}a^4 + \frac{1}{1....6}a^6 - \frac{1}{1....8}a^8 + \text{\&c.}$

Thus, if $n=10$, $\frac{1}{n}(p+p'+p''+p'''+p^{iv}) = \frac{50521}{1....10}$ = coefficient of a^{10}.

3. Let $\begin{cases} \text{Cos.} = 1 + ax^2 + bx^4 + cx^6 + \text{\&c.} \\ \text{Sec.} = 1 + Ax^2 + Bx^4 + Cx^6 + \text{\&c.} \end{cases}$

or $\begin{cases} \text{Tan.} = x + ax^3 + bx^5 + cx^7 + \text{\&c.} \\ \text{Cot.} = \frac{1}{x} + Ax + Bx^3 + Cx^5 + \text{\&c.} \end{cases}$

Then, because each product equal 1, we have,

$A = -a.$
$B = -(b + Aa).$
$C = -(c + Ab + Ba).$
$D = -(d + Ac + Bb + Ca).$
&c. &c.

Lastly, because the segment, contained by the arc a and its chord, is the difference of the sector and triangle upon them,

151. $\text{Segm.}a = \frac{1}{2}a - \frac{1}{2}s.$

$$= \tfrac{1}{2}\left(\frac{a^2}{1.2.3} - \frac{a^5}{1\ldots5} + \frac{a^7}{1\ldots7} - \frac{a^9}{1\ldots9} + \&c.\right)$$

TO EXPRESS AN ARC BY INFINITE SERIES, IN TERMS OF ITS SINE, TANGENT, &c.

To reverse the preceding series, in order to find the arc from its sine, tangent, &c. we have the following algebraical theorems, viz.:

152. If $z = x + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$, then

$$x = z - bz^2 + (2b^2 - c).z^3 + (5bc - 5b^3 - d).z^4 + \&c.$$

$$(14b^4 - 21b^2c + 6bd + 3c^2 - e).z^5 + (84b^3c - 42b^5 - 28b^2d - 28bc^2 + 7be + 7cd - f).z^6 + \&c.$$

153. If $z = x + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11} + \&c.$, then

$$x = z - bz^3 + (3b^2 - c)z^5 + (8bc - 12b^3 - d).z^7 + (55b^4 - 55b^2c + 10bd + 5c^2 - e).z^9 + \&c.$$

Now, let s, t, v denote the sine, tangent, and versed sine of the arc a; then by theorem [153,] reversing the series [145, 147,] we find,

154. $a = s + \frac{1}{2}\cdot\frac{1}{3}s^3 + \frac{1.3}{2.4}\cdot\frac{1}{5}s^5 + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7}s^7 + \frac{1.3.5.7}{2.4.6.8}\cdot\frac{1}{9}s^9 + \&c.$

155. $a = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \frac{1}{11}t^{11} + \&c.$

156. $a = \sqrt{2v}.(1 + \frac{1}{2.3.2}v + \frac{1.3}{2.4.5.2^2}v^2 + \frac{1.3.5}{2.4.6.7.2^3}v^3 + \frac{1.3.5.7}{2.4.6.8.9.2^4}v^4 + \&c.)$

Also, by the same theorem, reversing the series 143, 144, putting tan. $(45 + a) - 1 = p$, and $1 - \text{tan.}\,(45 - b) = q$,

157. $a = \frac{1}{2}p - \frac{1}{2.2}p^2 + \frac{1}{3.4}p^3 - \frac{1}{5.8}p^4 + \frac{1}{6.8}p^6 - \frac{1}{7.16}p^7 + \frac{1}{9.32}p^9 - \frac{1}{10.32}p^{10} + \&c.$

158. $b = \frac{1}{2}q + \frac{1}{2.2}q^2 + \frac{1}{3.4}q^3 - \frac{1}{5.8}q^5 - \frac{1}{6.8}q^6 - \frac{1}{7.16}q^7 + \frac{1}{9.32}q^9 + \frac{1}{10.32}q^{10} + \&c.$

Hence, supposing $p = q$,

159. * $(b - a) = \frac{1}{2}p^2 - \frac{1}{3.8}p^6 + \frac{1}{5.32}p^{10} - \frac{1}{7.128}p^{14} + \frac{1}{9.512}p^{18} - \&c.$

* Comparing 159 with 155, we have this geometrical theorem:

Let $(45 - b)$, 45, $(45 + a)$, be three arcs, whose tangents are in arithmetical progression, having their common difference $= p$; then $\frac{1}{2}p^2 =$ rad. . tan. $(b - a) =$ the rectangle under the radius, and the cotan. sum of the extreme arcs.

160. $(b+a) = p + \frac{1}{3.2}p^3 - \frac{1}{5.4}p^5 - \frac{1}{7.8}p^7 + \frac{1}{9.16}p^9 + \frac{1}{11.32}p^{11} - \&c.$

161. $a = t - \frac{1}{3}t^3\frac{1}{1+t^2} - \frac{2}{3.5}t^5\frac{1}{(1+t^2)^2} - \frac{2.4}{3.5.7}t^7\frac{1}{(1+t^2)^3} - \&c.$

162. $a = \frac{t}{1+t^2} + \frac{2}{3}\cdot\frac{t^3}{(1+t^2)^3} + \frac{2.4}{3.5}\cdot\frac{t^5}{(1+t^2)^5} + \frac{2.4.6}{3.5.7}\cdot\frac{t^7}{(1+t^2)^4} + \&c.$

To express a segment of a circle by infinite series, in terms of the sine of its arc, we have, [151, 154,]

163. Segm. $a = \frac{1}{2}\left(\frac{1}{2.3}s^3 + \frac{1.3}{2.4.5}s^5 + \frac{1.3.5}{2.4.6.7}s^7 + \frac{1.3.5.7}{2.4.6.8.9}s^9 + \&c.\right)$

To express it in terms of the versed sine of its half, we have,

164. Segm. $2a = \sqrt{(2v)}\cdot\left(\frac{1}{3.2^{-2}}v - \frac{1}{2.5.2^{-1}}v^2 - \frac{1}{2.4.7.2^0}v^3 - \frac{1.3}{2.4.6.9.2}v^4 - \&c.\right)$

For the segment BAG (see fig. 1.)

$= 2$ segm. AB + triangle BAG

$=$ arc AB $-$ sin. BE [148.] + AE.EB

$= a - \sqrt{\left(1-(1-v)^2\right)} + v\sqrt{\left(1-(1-v)^2\right)}$

$= a - (1-v)\sqrt{(2v-v^2)}$

$= a - \sqrt{(2v)}\cdot\sqrt{(1-\frac{1}{2}v+2v^2-\frac{1}{2}v^3)}$

To express it in terms of the cosine of its half.

165. Segm. $2a$ = semicircle $- 2c\left(1 - \frac{1}{2.3}c^2 - \frac{1}{2.4.5}c^4 - \frac{1.3}{2.4.6.7}c^6 - \frac{1.3.5}{2.4.6.8.9}c^8 - \&c.\right)$

For the segment BAG

$=$ sem. QAN $-$ (CQ $+$ BE) . CE $-$ 2 segm. BD

$=$ sem. $-\left(1 + \sqrt{(1-c^2)}\right)c -$ ar. BD $+ c$

$=$ sem. $-$ ar. BD $- c\sqrt{(1-c^2)}$;

then, expressing the arc BD in terms of its sine c, the result is obtained.

TO EXPRESS THE CIRCUMFERENCE OF A CIRCLE IN PARTS OF THE RADIUS.

From any one of the preceding series', [151—155.] the circumference of a circle may be computed in parts of the radius to any degree of accuracy. But as these series converge swiftly, only when the arc is small, it is requisite to find a very small part of the circumference that has a given sine, tangent, &c.; or, several very small arcs, which, connected together by the signs $+$ and $-$, make a known part of the circumference.

Of these methods, the latter is that which is most easily effected.

Thus, if $\frac{1}{4} = \text{sin. } a$; then [88.] sin. $3a = \frac{11}{16}$, and [89.] cos. $6a$, or sin. $(90 - 6a) = \frac{7}{128}$: Therefore, putting $s = \frac{1}{4}$, and finding the sum of the series 151; also putting $s = \frac{7}{128}$, and finding the sum of the series again, six times the former sum added to the latter will give the length of a quadrant.

Again, if tan. $(45 - a) = \frac{1}{2}$, then [4.] tan. $(45 + a) = 2$, and [18, 19.] tan. $2a = \frac{3}{4}$: Suppose tan. $(45 - b) = \frac{5}{4}$, then $2a + b = 45°$. Hence the sum of the series 155, when $q = \frac{1}{2}$, added to double the sum of the same series, when $q = \frac{1}{4}$, will give the length of a quadrant.

But this method will be best illustrated in the application of series 152, whose terms are the most simple. The tan. $45°$ $(= 1)$ is the only tangent of a known part of the circumference that has been found expressed in rational numbers; but, if $t = 1$, the series will converge so slowly as to be absolutely unfit for the purpose. As any arc, however, having a given tangent, may be resolved into other arcs having given tangents, we may find fractional values of t to answer our purpose, in the following manner :

163. Let $\frac{a}{b}$, $\frac{c}{d}$ be tangents of two arcs, of which the former is the greater.

The tangent of their sum is, [51.] ..... $\frac{ad + bc}{bd \smile ac}$.

The tangent of their difference is, [52.] ..... $\frac{ad - bc}{ac + bd}$.

The tangent of double the former is, [53.] $\frac{2ab}{a^2 \smile b^2}$.

The tangent of the difference between the former and 45° is, [54.] ..... $\frac{a \smile b}{a + b}$.

Now, if $\frac{1}{c}$ be the given tangent of an arc, and it be required to find $\frac{1}{x}$, $\frac{1}{y}$, the tangents of two other arcs whose sum is equal to the former, so that x and y may be whole numbers, we have $\frac{1}{c} = \frac{x + y}{xy - 1}$; therefore, $cx + cy = xy - 1$, and $cx + 1 = xy - cy = (x - c)y$. Suppose $x - c = n$, then $cx + 1 = ny$, $x = c + n$; whence $ny = c^2 + nc + 1$, and $y = c + \frac{c^2 + 1}{n}$.

164. Thus, if $c^2 + 1 = mn$, then $\frac{1}{c + m}$ and $\frac{1}{c + n}$

are the tangents of two arcs, of which $\frac{1}{c}$ is the tangent of their sum.

165. In like manner, if $c^2 + 1 = mn$, then $\frac{1}{c - n}$ and $\frac{1}{m - c}$ are the tangents of two arcs, of which $\frac{1}{c}$ is the tangent of their difference.

First, then, let $\frac{1}{c} = \text{tan. } 45^\circ = 1$; $c = 1$, and $c^2 + 1 = 2 = 1 \cdot 2$; therefore, [164.] $\frac{1}{2}$, $\frac{1}{3}$ are tangents of two arcs whose sum is 45°.

Next, let $\frac{1}{c} = \frac{1}{2}$, then $c^2 + 1 = 5 = 1 \cdot 5$; therefore, [164.] $\frac{1}{3}$, $\frac{1}{7}$ are the tangents of two arcs whose sum has $\frac{1}{2}$ for its tangent.

166. Hence, $45^\circ = 2\overline{\left|\frac{1}{3}\right.} + \overline{\left|\frac{1}{7}\right.}$ = double the arc whose tangent is $\frac{1}{3}$, added to the arc whose tangent is $\frac{1}{7}$.

Also, taking $\frac{1}{4}$ as the tangent of an arc, $\frac{8}{15}$ is the tangent of its double, [163.] $\frac{47}{52}$ the tangent of its triple, [163.] and $\frac{5}{99}$ the tangent of the excess of 45° above the triple; but $\overline{\left|\frac{5}{99}\right.} = \overline{\left|\frac{1}{20}\right.} + \overline{\left|\frac{1}{1985}\right.}$; therefore, $45^\circ = 3\overline{\left|\frac{1}{4}\right.} + \overline{\left|\frac{1}{20}\right.} + \overline{\left|\frac{1}{1985}\right.}$, &c.

Proceeding in this manner, we may resolve 45° se-

veral ways into small arcs, having given tangents, as in the following examples:

$$45^\circ = 3\left|\frac{1}{7}\right. + 4\left|\frac{1}{68}\right. + 2\left|\frac{1}{117}\right. = 3\left|\frac{1}{9.5}\right. + 7\left|\frac{1}{27}\right. + 2\left|\frac{1}{1068}\right.$$

$$45^\circ = 4\left|\frac{1}{5}\right. - \left|\frac{1}{239}\right. \qquad = 3\left|\frac{1}{10}\right. - 4\left|\frac{1}{515}\right. - \left|\frac{1}{239}\right.$$

167. Now, taking the last of these, viz.

$$45^\circ = 3\left|\frac{1}{10}\right. - 4\left|\frac{1}{515}\right. - \left|\frac{1}{239}\right.,$$

and putting $t = \frac{1}{10}$, $\frac{1}{515}$, and $\frac{1}{239}$ successively, we compute the circumference of a circle in parts of its radius from series [155.], as follows:

AFFIRMATIVE TERMS.	NEGATIVE TERMS.
t =.8	$\frac{8}{3}t^3$ .00266, 66666, 66666, 66666, 7
t^5 =.00001, 6	$\frac{8}{7}t^7$ ....., 01142, 85714, 28571, 4
t^9 =....., 00008, 88888, 88888, 9	$\frac{8}{11}t^{11}$....., ..., 07272, 72727, 3
t^{13} =....., ..., 00061, 53846, 2	$\frac{8}{13}t^{15}$....., ..., ..., 53333, 3
t^{17} =....., ..., ..., 00470, 6	$\frac{8}{19}t^{19}$....., ..., ..., 4, 2
t^3 =....., 00097, 61511, 03310, 0	$4t^{\wedge}$ .00776, 69902, 91262, 13592, 2
t^7 =....., ..., ..., 00005, 9	$\frac{4}{5}t^{\wedge 5}$....., ..., 00022, 08278, 5
$t^{\wedge 3}$=....., 00244, 16591, 78708, 4	$t^{\wedge\wedge}$ .00418, 41004, 18410, 04184, 1
$t^{\wedge 7}$=....., ..., ..., 00320, 7	$\frac{1}{5}t^{\wedge\wedge}$....., ..., 00256, 47231, 4
+.80001, 60350, 67053, 25550, 7	—.01461, 78716, 69604, 94589, 1
—.01461, 78716, 69604, 94589, 1	
5° =.78539, 81633, 97448, 30961, 6	

Hence, half the circumference of a circle whose radius is 1, or the whole circumference of a circle whose diameter is 1, is nearly 3.14159,26535,89793,23846.

168. The same may be also very easily computed by the series 162, which converges more quickly. Thus, since [166.] $45^\circ = 2|\frac{1}{3} + |\frac{1}{7} = 2|\frac{1}{7} + 2|\frac{2}{11}$, we have

$$45^\circ = \frac{42}{100} + \frac{56}{10{,}000} + \frac{896}{10{,}000{,}000} + \frac{1536}{1{,}000{,}000{,}000} + \&c.$$
$$+ \frac{352}{1000} + \frac{22{,}528}{3{,}000{,}000} + \frac{256}{10{,}000}\,b + \frac{192}{7000}\,c + \frac{256}{9000}\,d + \&c.$$

Or,

$$45^\circ = \frac{42}{100} + \frac{2}{3}a \cdot \frac{2}{100} + \frac{4}{5}\,b \cdot \frac{2}{100} + \frac{6}{7}\,c \cdot \frac{2}{100}$$
$$+ \frac{8}{9}\,d \cdot \frac{2}{100} + \&c.$$
$$+ \frac{352}{1000} + \frac{2}{3}a' \cdot \frac{32}{1000} + \frac{4}{5}b' \cdot \frac{32}{1000} + \frac{6}{7}c' \cdot \frac{32}{1000} + \&c.$$

where a, b, c, &c. also a', b', c', &c. denote each the whole of the preceding term.

169. As the decimal subdivision of the quadrant is undoubtedly preferable to the sexagesimal, in the following method of constructing a Table of Sines, we shall suppose the quadrant divided into 90 degrees, but

each of these, as was proposed by Briggs, subdivided into 100 minutes, or into 10,000 seconds.

Putting p for the length of 45° to the radius 1, [167.] we have the length of an arc of 12° $= \frac{12}{45} p$ $=$ .20943,95102,39319,549; and by substituting this number for a in the series [145, 146.], we find the sine and cosine of 12° as follows:

AFFIRMATIVE TERMS.	NEGATIVE TERMS.
a = .20943, 95102, 39319, 549	$\frac{1}{6}a^3$ = .00153, 11741, 57052, 541
$\frac{1}{1\ldots5}a^5$ = ..., 33582, 40710, 951	$\frac{1}{1\ldots7}a^7$ = ..., 00035, 07355, 270
$\frac{1}{1\ldots9}a^9$ = ..., ..., 02136, 803	$\frac{1}{1\ldots11}a^{11}$ = ..., ..., ..., 852
+ .20944, 28684, 82167, 303	— .00153, 11776, 64408, 663
— .00153, 11776, 64408, 663	
Sin. 12° = .20791, 16908, 17758, 640	
$\frac{1}{1\ldots4}a^4$ = .00008, 01720, 91386, 047	$\frac{1}{2}a^2$ = .02193, 24542, 24653, 019
$\frac{1}{1\ldots8}a^8$ = ..., ..., 91822, 346	$\frac{1}{1\ldots6}a^6$ = ..., 01172, 24714, 961
	$\frac{1}{1\ldots10}a^{10}$ = ..., ..., 00044, 753
+ 1.00008, 01721, 83208, 393	— .02193, 25715, 49412, 733
— .02193, 25715, 49412, 733	
Cos. 12° = .97814, 76006, 33795, 660	

170. By dividing each of these terms by the same power of 2 as of a, and then collecting the quotients, we get the sine and cosine of 6°. Also, by dividing each of the terms by the same power of 3 as of a, we find, in like manner, the sine and cosine of 4°.

From these quotients again, by a similar process, we obtain the sine and cosine of 3°, 1°.5; 2°, 1°, 0°.5. Whence, multiplying by the powers of 10, we have those of 15°, 10°, 5°; and multiplying by the powers of 2, 3, those of 8°, 9°.

171. Lastly, the sines and cosines of 7°, 11°, 13°, and 14° may be obtained from those already found by the theorems 55—58.

Thus, if C = cos. 1°,

Sin. 7° = 2C . sin. 6° — sin. 5°,

Cos. 7° = 2C . cos. 6° — cos. 5°, &c.

Then the accuracy of the whole computation may be tried, by finding the sine and cosine of 15° in this manner:

$$\text{Sin. } 15^\circ = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}), \text{ and}$$

$$\text{Cos. } 15^\circ = \text{sin. } 15^\circ + \tfrac{1}{2}\sqrt{2}.$$

172. Having thus found the sines and cosines of the first 15 whole degrees, and, at the same time, those of

as many tenths, hundredths and thousands, (there being no trouble in dividing by the powers of 10); the table may be filled up to one-third of the whole, by the differential method.

Thus, if A and A + na be two arcs whose sines and cosines are found, and it be required to find those of the intermediate arcs A + a, A + $2a$, A + $3a$, &c. let s, c be the sine and cosine of A; then

$$\text{Sin. A} = s.$$
$$\text{Sin. A} + a = s + ac - \tfrac{1}{2}a^2s - \tfrac{1}{6}a^3c.$$
$$\text{Sin. A} + 2a = s + 2ac - \tfrac{4}{2}a^2s - \tfrac{8}{6}a^3c.$$
$$\text{Sin. A} + 3a = s + 3ac - \tfrac{9}{2}a^2s - \tfrac{27}{6}a^3c.$$

I. DIFFERENCE.	II. DIFFERENCE.	III. DIFFERENCE.
$ac - \frac{1}{2}a^2s - \frac{1}{6}a^3c$	$-a^2s - a^3c$	
$ac - \frac{3}{2}a^2s - \frac{7}{6}a^3c$	$-a^2s - 2a^3c$	$-a^3c.$
$ac - \frac{6}{2}a^2s - \frac{19}{6}a^3c$		

Now, supposing $a = 0^{\circ}.001$, the length of that arc in parts of the radius, and the several powers thereof, will be as follows:

a = . 00001, 74532, 92519, 94329, 57692, 37
a^2 = . ——, 00003, 04617, 41978, 67085, 99
a^3 = . ——, ——, 00005, 31657, 65888, 65
a^4 = . ——, ——, ——, 00009, 27917, 74
a^5 = . ——, ——, ——, ——, 00016, 19
&c. &c.

From which the first term of each order of differences is to be calculated; then, supposing the third difference equal, in respect to the limits of the table, the whole of the second order will be obtained from its first term by the continual addition of the third difference. The whole of the first order will next be obtained from its first term, by the continual subtraction of the successive differences of the second order. And, lastly, the series of sines will be obtained from its first term, by the continual addition of the successive differences of the first order.

If we take $n = 100$, it is evident, from the above table of the powers of a, that these three orders of differences will be sufficient to give the sines true to, at least, 12 places of decimals.

If $n = 1000$, the sines found from the same differences will be true to 8 or 9 places.

173. One third of the table being thus constructed, the sines of the following 16 arcs in whole degrees, viz.

$37°, 38°, 39° \ldots . 44°, 46° \ldots . . 51°, 52°, 53°,$

may be computed by some of the preceding theorems, as, if c, s denote the sine and cosine of a ($1° \ldots . 8°$) found before; then

$$\text{Sin. } (45 + a) = (c + s) \cdot \tfrac{1}{2}\sqrt{2}.$$
$$\text{Sin. } 45 - a = (c - s) \cdot \tfrac{1}{2}\sqrt{2}.$$

Where $\frac{1}{2}\sqrt{2} = .70710\ 67811\ 86547$, &c. of which, if a table of multiples from 1 to 100 be formed, the computation will be much facilitated.

All the remaining arcs of the quadrant in whole degrees may then be obtained from those already computed, by simple addition and subtraction, by means of the following theorems:

THEOREM 1.

$$\text{Sin. } a + \text{sin. } (60 - a) = \text{sin. } (60 + a).$$

THEOREM 2.

$$\text{Sin. } a + \text{sin. } (36 - a) + \text{sin. } (72 + a) = \text{sin. } (36 + a) + \text{sin. } (72 - a).$$

By theorem 1. we have the sines of

23°, 22°, 21° . . . . . 16°, 74°, 73° . . . . . 69°, 68°, 67°.

By theo. 2. we have those of 24°, 35°, 34°, 33°, 32°, 31°, 66°.

By theo. 1. ——— 36°, 25°, 26°, 27°, 28°, 29°, 54°.

By theo. 2. ——— 61°, 62°, 63°, 64°, 65°.

By theo. 1. ——— 59°, 58°, 57°, 56°, 55°.

These two-thirds of the table are then to be filled up as the first third, by the differential method.

174. We shall add the following table of verification, in which the sines of every multiple of 3° in the quadrant may be otherwise computed:

Putting

$a = \frac{1}{2}\sqrt{2}$.
$b = \frac{1}{2}\sqrt{3}$.
$c = \frac{1}{4}(\sqrt{5}-1)$.
$d = \frac{1}{4}(\sqrt{5}+1)$.
$e = \frac{1}{4}(\sqrt{6}+\sqrt{2})$.
$f = \frac{1}{4}(\sqrt{6}-\sqrt{2})$.
$g = \frac{1}{8}(\sqrt{10}+\sqrt{2})$.
$h = \frac{1}{8}(\sqrt{10}-\sqrt{2})$.
$k = \frac{1}{8}(\sqrt{15}+\sqrt{3})$.
$l = \frac{1}{8}(\sqrt{15}-\sqrt{3})$.
$m = \frac{1}{4}\sqrt{(5+\sqrt{5})}$.
$n = \frac{1}{4}\sqrt{(5-\sqrt{5})}$.
$o = \frac{1}{8}\sqrt{(10+2\sqrt{5})}$.
$p = \frac{1}{8}\sqrt{(10-2\sqrt{5})}$.
$q = \frac{1}{8}\sqrt{(30+6\sqrt{5})}$.
$r = \frac{1}{8}\sqrt{(30-6\sqrt{5})}$.

TABLE.

Sin. 6° $= r - \frac{1}{2}d$.
Sin. 9° $= g - n$.
Sin. 12° $= o - l$.
Sin. 15° $= f$.
Sin. 18° $= c$.
Sin. 24° $= k - p$.
Sin. 27° $= m - h$.
Sin. 30° $= \frac{1}{2}$.
Sin. 36° $= 2p$.
Sin. 42° $= q - \frac{1}{2}c$.
Sin. 45° $= a$.
Sin. 48° $= o + l$.
Sin. 54° $= d$.
Sin. 60° $= b$.
Sin. 63° $= m + h$.
Sin. 66° $= r + \frac{1}{2}d$.
Sin. 72° $= 2o$.
Sin. 75° $= e$.
Sin. 78° $= q + \frac{1}{2}c$.
Sin. 81° $= g + n$.
Sin. 84° $= k + p$.

Sin. $3° = \sqrt{\frac{1 - \text{sin. } 84}{2}}$.	Sin. $21° = \text{sin. } 81° - \text{sin. } 39°$.
Sin. $87° = \sqrt{\frac{1 + \text{sin. } 84}{2}}$.	Sin. $33° = \text{sin. } 87° - \text{sin. } 27°$.
Sin. $39° = \sqrt{\frac{1 - \text{sin. } 12}{2}}$.	Sin. $57° = \text{sin. } 63° - \text{sin. } 3°$.
Sin. $51° = \sqrt{\frac{1 + \text{sin. } 12}{2}}$.	Sin. $69° = \text{sin. } 51° + \text{sin. } 9°$.

175. Algebra furnishes the following rule for extracting the square root of numbers by a very swiftly converging series, viz.

$$\sqrt{(a^2 + r)} = a + \frac{r}{2a} - \frac{r^2}{4ab} - \frac{r^4}{8abc} - \frac{r^6}{16abcd} \text{ \&c.}$$

Where $b = (2a^2 + r)$, $c = (2b^2 - r^2)$, $d = (2c^2 - r^4)$, $e = (2d^2 - r^6)$, &c.

EXAMPLE.

To find $\sqrt{10}$; put $a = 19$, then $r = -1$; and,

$$6\sqrt{10} = \sqrt{(19^2 - 1)} = 19 - \frac{1}{38} - \frac{1}{38.1442} - \frac{1}{38.1442.2079364} - \text{\&c.}$$

$$\text{or, } \sqrt{10} = \frac{19}{6} - \frac{1}{6.38} - \frac{1}{6.38.1442} - \frac{1}{6.38.1442.2079364} - \text{\&c.}$$

176. Since the invention of logarithms, the natural tangents and secants are seldom introduced into trigonometrical calculations; we shall not, therefore, insist particularly on the construction of these, but only observe, that if half the table of tangents be (by methods similar to the foregoing) constructed, the rest of the tangents and all the secants may be obtained, by simple addition and subtraction, [16, 17.] from the tangents found.

OF LOGARITHMS.

177. LOGARITHMS are the measures of ratios. Every quantity is measured by another of the same kind, termed the measuring Unit. Thus, a line is measured by a line, an angle by an angle, and a ratio by a ratio. The number of measuring units which a quantity contains is the measure of that quantity. No measuring unit, however small, will exactly measure all other quantities of the same kind. But, the smaller the measuring unit is, or the more it is subdivided, the more accurately will the measures of all quantities of the same kind be ascertained.

Let, then, the measuring unit of ratios be expressed by the ratio $1 : (1 + x)^r$, which may itself be subdivided into an indefinitely great number (r) of very small ratios, each equal to $1 : (1 + x)$, x being indefinitely small: And from the nature of ratios,

$$\text{The logarithms of } \begin{cases} 1 : (1+x)^r \\ 1 : (1+x)^{2r} \\ 1 : (1+x)^{3r} \\ 1 : (1+x)^{\lambda r} \\ \text{\&c.} \end{cases} \text{ are } \begin{cases} 1 \\ 2 \\ 3 \\ \lambda \\ \text{\&c.} \end{cases}$$

Now, let λ be the logarithm of the given ratio $1 : (1 + q)$, or simply the logarithm of $(1 + q)$, (when it is so expressed the antecedent term of the ratio is always understood to be 1,) and let it be required to find λ in terms of q, and *vice versâ*.

Because λ is the log. $(1 + x)^{\lambda r}$, putting $\lambda r = n$, we have,

$$1 + x = (1+q)^{\frac{1}{n}} = 1 + \frac{1}{n}q - \frac{1}{n}\cdot\frac{n-1}{2n}q^2$$
$$+ \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}q^3 - \text{\&c.};$$

but n being so great a number that it cannot be sensibly increased or diminished by a single unit, we may

reject 1 in the numerators of the fractions $\frac{n-1}{2n}$, $\frac{2n-1}{3n}$, &c. then we have,

$$1+x=1+\frac{1}{n}\cdot q-\frac{1}{n}\cdot\tfrac{1}{2}q^2+\frac{1}{n}\cdot\tfrac{1}{3}q^3-\frac{1}{n}\cdot\tfrac{1}{4}q^4$$
$$+\frac{1}{n}\cdot\tfrac{1}{5}q^5-\text{&c.}$$

or, $nx=\lambda rx=q-\frac{1}{2}q^2+\frac{1}{3}q^3-\frac{1}{4}q^4+\frac{1}{5}q^5-$ &c.

Now, x being supposed indefinitely small, and r indefinitely great, in respect to 1, but the particular relation between them being indeterminate, we may assume $rx=\frac{1}{m}$ a given quantity; and we shall have this general expression for the value of λ, viz.

178. Log. $(1+q)=m(q-\frac{1}{2}q^2+\frac{1}{3}q^3-\frac{1}{4}q^4+\frac{1}{5}q^5-$ &c.)

Next, to find $1+q$ in terms of λ. By a similar process, or by reversing the preceding series, we have,

$$179.\quad 1+q=1+\frac{\lambda}{m}+\frac{1}{2}\cdot\frac{\lambda^2}{m^2}+\frac{1}{2.3}\cdot\frac{\lambda^3}{m^3}$$
$$+\frac{1}{2.3.4}\cdot\frac{\lambda^4}{m^4}+\text{&c.}$$

According to the different values assigned to m, there may be as many different systems of logarithms formed. Hence m is termed the modulus of the sys-

tem. The system most easily computed appears to be that in which the modulus $m = 1$. And the methods of computation in this system (termed hyperbolic) may be extended to any other system of which the modulus is known. Supposing, then, $m = 1$, we have

Hyp. log. $(1+q) = +q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \frac{1}{4}q^4 + \frac{1}{5}q^5 - \frac{1}{6}q^6 +$ &c.

Hyp. log. $(1-q) = -q - \frac{1}{2}q^2 - \frac{1}{3}q^3 - \frac{1}{4}q^4 - \frac{1}{5}q^5 - \frac{1}{6}q^6 -$ &c.

Therefore,

180. Hyp. log. $\frac{1+q}{1-q} = 2 \cdot (q + \frac{1}{3}q^3 + \frac{1}{5}q^5 + \frac{1}{7}q^7 + \frac{1}{9}q^9 +$ &c.)

Hence, supposing $\frac{1+q}{1-q} = \frac{a+d}{a}$, then $q = \frac{d}{2a+d}$, and $\frac{1}{1-q^2} = \frac{(a+\frac{1}{2}d)^2}{a(a+d)}$, or $\frac{(2a+d)^2}{4a(a+d)}$.

181. $\frac{1}{2}$ hyp. log. $\frac{a+d}{a} = \frac{d}{2a+d} + \frac{1}{3} \cdot \frac{d^3}{(2a+d)^3} + \frac{1}{5} \cdot \frac{d^5}{(2a+d)^5} + \frac{1}{7} \cdot \frac{d^7}{(2a+d)^7} +$ &c.

182. Hyp. log. $\frac{(2a+d)^2}{4a(a+d)} = \frac{d^2}{(2a+d)^2} + \frac{1}{2} \cdot \frac{d^4}{(2a+d)^4} + \frac{1}{3} \cdot \frac{d^6}{(2a+d)^6} +$ &c.

By which two series, the log. a being given, those of $a + d$ and $2a + d$ may be found.

Again, putting λ for the log. $\frac{a}{b}$, a being greater than b,

$$\frac{a}{b} = 1 + \lambda + \frac{1}{2}\lambda^2 + \frac{1}{2.3}\lambda^3 + \frac{1}{2.3.4}\lambda^4 + \frac{1}{2.3.4.5}\lambda^5 + \&c.$$

183. $$\frac{b}{a} = 1 - \lambda + \frac{1}{2}\lambda^2 - \frac{1}{2.3}\lambda^3 + \frac{1}{2.3.4}\lambda^4 - \frac{1}{2.3.4.5}\lambda^5 + \&c.$$

Hence, $\frac{a}{b} - \frac{b}{a} = \frac{a^2 - b^2}{ab}$, and therefore

184. $$\frac{a^2 - b^2}{2ab} = \lambda + \frac{1}{2.3}\lambda^3 + \frac{1}{2.3.4.5}\lambda^5 + \frac{1}{2.3.4.5.6.7}\lambda^7 + \&c.$$

Which series reversed, (putting $\frac{a^2 - b^2}{2ab} = z$), gives,

185. $$\lambda = \log. \frac{a}{b} = z - \frac{1}{2.3}z^3 + \frac{1.3}{2.4.5}z^5 - \frac{1.3.5}{2.4.6.7}z^7 + \frac{1.3.5.7}{2.4.6.8.9}z^9 - \&c.$$

$$= \log. \frac{1+q}{1-q}, \text{ when } z = \frac{2q}{1-q^2}.$$

186. In these hyperbolic logarithms, the measuring ratio is

$$1 : (1 + x)^r, \text{ or } 1 : 2.71828,18284, \&c.$$

as may be found from the series [179], putting $m = 1$, and $\lambda = 1$. But, as 10 is the root of the arithmetical scale, so the ratio 1 : 10 is the most convenient measuring unit; and the logarithms in common use have

been constructed on this principle. The common logarithm of 10, then, being 1 to the modulus m, those of

1000, 100, 10, 1, 0.1, 0.01, 0.001, &c.

are 3, 2, 1, 0, -1, -2, -3, &c.

But, com. log. 10 : hyp. log. 10 : : m : 1 : : 1 : $\frac{1}{m}$,

[178]. Hence $m = \frac{1}{\text{hyp. log. } 10}$; that is, the modulus of the common system is the reciprocal of the hyp. log. 10.

FIRST METHOD OF CONSTRUCTING A TABLE OF COMMON LOGARITHMS.

187. The first thing to be done is to find the hyp. log. 10. But the series above [180, 181.] being only adapted to finding the logarithms of those ratios (or fractions) whose terms are nearly equal, we must resolve the ratio 1 : 10 into its component parts, and find its logarithm in a manner similar to that in which we determined the circumference of the circle. Now, since $10 = 2 \times 2 \times 2 \times \frac{5}{4}$, or the ratio of 1 : 10 is compounded of thrice that of 1 : 2 and of 4 : 5, the

log. (1 : 2), or putting $a = 1$, and $d = 1$, may be found from the fraction $\frac{1}{3}$, [181.], and the log $\frac{5}{4}$ from the fraction $\frac{1}{9}$.

Thus $\frac{1}{2}$ log. $10 = 2\left|\frac{1}{3} + \left|\frac{1}{9} = 3\right.\right.$ times the value of the series [181.] when $\frac{d}{2a+d} = \frac{1}{3}$, added to the value of the same when $\frac{d}{2a+d} = \frac{1}{9}$.

In order to resolve these fractions into smaller ones, let M, N be two numbers, and $\frac{a}{b}, \frac{c}{d}$ the fractions from which their logarithms are to be computed; that is, [180.], let $\frac{M-1}{M+1} = q = \frac{a}{b}$, &c.;

then,

188.
$$\begin{cases} \frac{1}{2}\text{log. } M = \left|\frac{a}{b}, \ \frac{1}{2}\text{log. } M^2 = \left|\frac{2ab}{a^2+b^2}\right.\right. \\ \frac{1}{2}\text{log. } M^3 = \left|\frac{a^3+3ab^2}{3a^2b+b^3}\right., \text{ &c.} \\ \frac{1}{2}\text{log. } MN = \left|\frac{a}{b} + \left|\frac{c}{d} = \left|\frac{ad+bc}{bd+ac}\right.\right.\right. \\ \frac{1}{2}\text{log. } \frac{M}{N} = \left|\frac{a}{b} - \left|\frac{c}{d} = \left|\frac{ad-bc}{bd-ac}\right., \text{ &c.}\right.\right. \end{cases}$$

Now, let it be required to resolve the fraction $\frac{1}{a}$ into two others $\frac{1}{x}, \frac{1}{y}$, so that $\left|\frac{1}{a} = \left|\frac{1}{x} + \left|\frac{1}{y}\right.\right.\right.$. We have,

[188.], $\frac{1}{a} = \frac{x+y}{xy+1}$, $xy + 1 = ax + ay$, and $x(y-a) = ay - 1$. Suppose $y - a = n$; then $x = \frac{an + a^2 - 1}{n}$.

189. Hence, if $a^2 - 1 = mn$, then

$$\left|\frac{1}{a}\right. = \left|\frac{1}{a+m}\right. + \left|\frac{1}{a+n}\right..$$

In like manner, to find $\frac{1}{x}$, $\frac{1}{y}$, so that $\frac{1}{a} = \left|\frac{1}{x}\right. - \left|\frac{1}{y}\right.$.

190. Put $a^2 - 1 = mn$, as before, then

$$\left|\frac{1}{a}\right. = \left|\frac{1}{a-m}\right. - \left|\frac{1}{n-a}\right..$$

But here m must always be less than $a - 1$; because x, or $a - m$ must be greater than 1.

Thus, if $\frac{1}{a} = \frac{1}{11}$, then $a^2 - 1 = 120 = 1 \cdot 120 = 2 \cdot 60 = 3 \cdot 40$, &c.

Therefore, $\left|\frac{1}{11}\right. = \left|\frac{1}{12}\right. + \left|\frac{1}{131}\right. = \left|\frac{1}{15}\right. + \left|\frac{1}{71}\right.$ = &c. [189.]

and $\left|\frac{1}{11}\right. = \left|\frac{1}{10}\right. - \left|\frac{1}{109}\right. = \left|\frac{1}{9}\right. - \left|\frac{1}{49}\right.$ = &c. [190.]

Again, if $\frac{1}{a} = \frac{3}{253}$, then $a = \frac{253}{3}$, $a^2 - 1 = \frac{64000}{9}$

$$= \frac{200}{3} \cdot \frac{320}{3} = \frac{10}{3} \cdot \frac{6400}{3} = \text{\&c. and}$$

$$\left|\frac{3}{253}\right. = \left|\frac{1}{151}\right. + \left|\frac{1}{191}\right. = \left|\frac{1}{81}\right. - \left|\frac{1}{2049}\right. = \text{\&c.}$$

Hence, $\frac{1}{2}$ log. 10

$$= {}^{8}\left|\tfrac{1}{3}\right. + \left|\tfrac{1}{9}\right. = {}^{10}\left|\tfrac{1}{9}\right. + {}^{3}\left|\tfrac{3}{283}\right. = {}^{10}\left|\tfrac{1}{9}\right. + {}^{3}\left|\tfrac{1}{131}\right. + {}^{3}\left|\tfrac{1}{191}\right. = \&c.$$

and log. 10

$$= {}^{20}\left|\tfrac{1}{9}\right. + {}^{6}\left|\tfrac{1}{131}\right. + {}^{6}\left|\tfrac{1}{191}\right. = {}^{20}\left|\tfrac{1}{9}\right. + {}^{12}\left|\tfrac{1}{191}\right. + {}^{6}\left|\tfrac{1}{721}\right.$$

191. From this last expression, the calculation to be made for the hyp. log. 10, true to 15 places of decimals, is as follows:

$\frac{20}{9}$	= 2.22222, 22222, 22222, 22		2.24943, 55131, 42097, 5
$\frac{1}{3}\cdot\frac{20}{9^3}$	.00914, 49474, 16552, 35	$\frac{12}{191}$	= .06282, 72251, 30890, 0
$\frac{1}{5}\cdot\frac{20}{9^5}$	. 6, 77403, 51233, 72	$\frac{1}{3}\cdot\frac{12}{191^3}$	. , 05740, 63440, 6
$\frac{1}{7}\cdot\frac{20}{9^7}$	. 05973, 57594, 65	$\frac{1}{5}\cdot\frac{12}{191^5}$	. , 09441, 5
$\frac{1}{9}\cdot\frac{20}{9^9}$	. 57, 35943, 98	$\frac{1}{7}\cdot\frac{12}{191^7}$	. , 1
$\frac{1}{11}\cdot\frac{20}{9^{11}}$	. , 57938, 82	$\frac{6}{721}$	.00832, 17753, 12066, 5
$\frac{1}{13}\cdot\frac{20}{9^{13}}$	. 605, 25	$\frac{1}{3}\cdot\frac{6}{721^3}$	. 53, 36102, 9
$\frac{1}{17}\cdot\frac{20}{9^{17}}$	. , 07	$\frac{1}{5}\cdot\frac{6}{721^5}$	. 6, 1
	2.24943, 55131, 42097, 54		

Hyp. log. 10 = 2.30258, 50929, 94045, 6

The reciprocal of which, viz. 0.43429,44819,03251,8 is the modulus of the common system.

Hence $2m$ = 0.86858,89638,06503,6.

192. Now to find the common log. 2, we have
$\frac{1}{2}$ hyp. log. $\frac{1024}{1000} = \left|\frac{5}{253}\right. = {}^{2}\left|\frac{1}{191}\right. + \left|\frac{1}{721}\right.$: Therefore
10 com. log. $2 = 3$ com. log. $10 + {}^{4m}\left|\frac{1}{191}\right. + {}^{2m}\left|\frac{1}{721}\right.$.
And to find the common log. 3, we have
$\frac{1}{2}$ h. log. $\frac{81}{80} = \left|\frac{1}{161}\right. = \left|\frac{1}{191}\right. + \left|\frac{1}{10225}\right.$: Therefore
4 com. log. $3 =$ com. log. $80 + {}^{2m}\left|\frac{1}{191}\right. + {}^{2m}\left|\frac{1}{10225}\right.$.

$\frac{4m}{191}$	0 . 00909, 51723, 95879, 6	c. l. 80	1 . 90308, 99869, 91943, 6
$\frac{1}{3}\cdot\frac{4m}{191^3}$	. 831, 04194, 8	${}^{2m}\left\|\frac{1}{191}\right.$	. 00454, 76277, 50720, 6
$\frac{1}{5}\cdot\frac{4m}{191^5}$	. 1366, 8	$\frac{2m}{1025}$	. 84, 74038, 67128, 3
$\frac{2m}{721}$	. 00120, 47003, 65889, 7	$\frac{1}{3}\cdot\frac{2m}{1025^3}$	2, 68857, 1
$\frac{1}{3}\cdot\frac{2m}{721^3}$	. 7, 72480, 0		0, 2
	0, 9		
	10)3 . 01029, 99566, 29811, 8		4)1 . 90848, 50188, 78649, 8
C. Log. 2	0 . 30102, 99956, 63981, 2	C. log. 3	0 . 47712, 12547, 19662, 4

From these the log. 19, and log. 7, are easily deduced.
For log. $19 = 6$ log. $2 +$ log. $3 -$ log. $10 - {}^{2m}\left|\frac{1}{191}\right.$.
3 log. $7 = 3$ log. $19 -$ log. $20 + {}^{2m}\left|\frac{1}{13719}\right.$.

193. For all other primes, the following rule, derived from the resolution of the logarithmic quantity $\left|\frac{1}{a}\right.$, may suffice, viz.:

$$\left|\frac{1}{a}\right. - {}^{2}\left|\frac{1}{2a}\right. = \left|\frac{1}{4a^2-3a}\right.,$$

that is, $\log. \frac{a+1}{a-1} = 2 \log. \frac{2a+1}{2a-1} + \frac{2m}{4a^3-3a}$,

as this last quantity may be always taken so small, that the first term of the series will be sufficient. Or, the rule may be more conveniently expressed thus:

$$\left|\frac{2}{a}\right. - \left|\frac{1}{a}\right. = \left|\frac{2}{a^3-3a}\right..$$

194. That is, $\log. \frac{a+2}{a-2} = 2 \log. \frac{a+1}{a-1} + \frac{4m}{a^3-3a}$.

An example will render the application of it easy. Let it be required to find the log. 7 from those of the inferior primes only. We must choose five numbers, differing by unit, of which, neglecting the middle number, one is a multiple of 7 by a number whose logarithm is given, and the other three have given logarithms. Such numbers are the following, 24, 25, 26, 27, 28; for, neglecting the middle number 26, one of the remaining four, viz. 28, is a multiple of 7 by 4, whose logarithm is given; and the other three, viz. 24, 25, 27 have given logarithms. By putting $a=26$, therefore, we get,

$$\text{Log.} \frac{28}{24} = 2 \log. \frac{27}{25} + \left|\frac{1}{8749}\right.^{2m}.$$

Hence,

$$\log. 7 = 7 \log. 3 + \log. 2 - 4 \log. 5 + \frac{2m}{8749} + \tfrac{1}{3} \cdot \frac{2m}{8749^3}.$$

195. In like manner, putting $a = 97$, we get,

$$\text{Log. } 11 = 2\log.98 - 2\log.96 + \log.95 - \log.9 + \frac{2m}{456191}.$$

$$a = 362.$$

$$\text{Log. } 13 = 2\log.363 - 2\log.361 + \log.360 - \log.28 + \frac{2m}{23718421}.$$

$$a = 287.$$

$$\text{Log. } 17 = \log.288 - \log.286 + \tfrac{1}{2}\log.285 + \frac{m}{23639042}.$$

$$a = 274.$$

$$\text{Log. } 23 = 2\log.275 - 2\log.273 + \log.272 - \log.12 + \frac{2m}{10285001}.$$

$$a = 727.$$

$$\text{Log. } 29 = \log.729 - 2\log.728 + 2\log.726 - \log.25 + \frac{2m}{192119201}.$$

&c. &c.

If the logarithm of each prime, as it is found, be entered in the Table, and also the logarithms of every multiple of that prime by the inferior numbers, the proper value of a for finding the next prime will immediately appear.

SECOND METHOD.

196. We proceed to explain a more expeditious method of constructing a Table of logarithms. It appears, from the exemplification of the preceding rule, that, when a is not less than 100,

$$\left|\frac{2}{a}\right. - \left|\frac{1}{a}\right. - \left|\frac{1}{a}\right. = \left|\frac{2}{a^2-3a}\right..$$

In like manner, we find,

$$\left|\frac{3}{a}\right. - \left|\frac{2}{a}\right. - \left|\frac{1}{a}\right. = \left|\frac{6}{a^2-7a}\right..$$

$$\left|\frac{4}{a}\right. - \left|\frac{3}{a}\right. - \left|\frac{1}{a}\right. = \left|\frac{12}{a^2-13a}\right..$$

$$\left|\frac{5}{a}\right. - \left|\frac{4}{a}\right. - \left|\frac{1}{a}\right. = \left|\frac{20}{a^2-21a}\right..$$

&c. &c.

$$\left|\frac{n}{a}\right. - \left|\frac{n-1}{a}\right. - \left|\frac{1}{a}\right. = \left|\frac{n.n-1}{a^2-(n^2-n+1)a}\right..$$

If then a be a very large number, (not less, suppose, than 100,000,) the first fraction above $\frac{2}{a^2-3a}$ will be a very small quantity, which being denoted by q, the succeeding ones, $\frac{6}{a^2-7a}$, $\frac{12}{a^2-13a}$, &c. will be very nearly equal to $3q$, $6q$, &c. Hence;

197.
$$\left|\frac{2}{a}\right. = {}^{3}\left|\frac{1}{a}\right. + q.$$
$$\left|\frac{3}{a}\right. = {}^{3}\left|\frac{1}{a}\right. + 4q.$$
$$\left|\frac{4}{a}\right. = {}^{4}\left|\frac{1}{a}\right. + 10q.$$
$$\left|\frac{5}{a}\right. = {}^{5}\left|\frac{1}{a}\right. + 20q.$$
$$\left|\frac{n}{a}\right. = {}^{n}\left|\frac{1}{a}\right. + \frac{n^3-n}{6}\,q.$$

198. From these equations, by equating the values of q, we deduce the following, viz. :

$$\left|\frac{3}{a}\right. = {}^{4}\left|\frac{2}{a}\right. - {}^{5}\left|\frac{1}{a}\right., \quad \left|\frac{4}{a}\right. = {}^{5}\left|\frac{3}{a}\right. - {}^{7}\left|\frac{1}{a}\right., \quad \left|\frac{5}{a}\right. = {}^{2}\left|\frac{4}{a}\right. - {}^{3}\left|\frac{1}{a}\right., \text{ \&c.}$$

How far these equations hold true, may be thus determined :

$$\left|\frac{3}{a}\right. = \frac{3}{a} + \frac{27}{3a^3} + \frac{243}{5a^5} + \frac{2187}{7a^7} + \text{\&c.}$$
$$-\,{}^{4}\left|\frac{2}{a}\right. = -\frac{8}{a} - \frac{32}{3a^3} - \frac{128}{5a^5} - \frac{1024}{7a^7} - \text{\&c.}$$
$$+\,{}^{5}\left|\frac{1}{a}\right. = \frac{5}{a} + \frac{5}{3a^3} + \frac{5}{5a^5} + \frac{5}{7a^7} + \text{\&c.}$$

$$* \quad * + \frac{24}{a^3} + \frac{240}{a^7} + \text{\&c.}$$

In this manner may be found innumerable rules of approximation, such as the following:

$$\text{Log.}\frac{a+3}{a-3} = 4\log.\frac{a+2}{a-2} - 5\log.\frac{a+1}{a-1} + \frac{48m}{a^5} + \&c.$$

$$6\ \text{Log.}\frac{a+4}{a-4} = 6\log.\frac{a+3}{a-3} - 14\log.\frac{a+2}{a-2} + 14\log.\frac{a+1}{a-1} + \frac{1440m}{a^7} + \&c.$$

&c. &c.

199. But, resuming the preceding series of logarithmic quantities [197.], and taking their differences, we have,

$$\begin{array}{lll|l|ll}
\text{Log.}\frac{a+1}{a-1} = \left|\frac{1}{a}\right. & & & & & \\
\text{Log.}\frac{a+2}{a-2} = {}^{2}\left|\frac{1}{a}\right. + q. & \left|\frac{1}{a}\right. + q. & & & & \\
\text{Log.}\frac{a+3}{a-3} = {}^{3}\left|\frac{1}{a}\right. + 4q. & \left|\frac{1}{a}\right. + 3q. & & 2q & & \\
\text{Log.}\frac{a+4}{a-4} = {}^{4}\left|\frac{1}{a}\right. + 10q. & \left|\frac{a}{a}\right. + 6q. & & 3q & q. & \\
\text{Log.}\frac{a+5}{a-5} = {}^{5}\left|\frac{1}{a}\right. + 20q. & \left|\frac{1}{a}\right. + 10q. & & 4q & q. &
\end{array}$$

&c. &c.

$$\text{Log.}\frac{a+n}{a-n} = {}^{n}\left|\frac{1}{a}\right. + \frac{n^2-n}{6}q.$$

From which we may find the logarithms of 1000 numbers, viz. from 99900 to 100900, by a very easy process. Since $\log. \frac{a+n}{a-n} = \overline{|\frac{n}{a}} = \frac{n}{a} + \frac{n^3}{3a^3} \quad \frac{n^5}{5a^5} + \&c.$ and the third term of the series is not included in the above logarithms, the number n must be so taken that $\frac{n^5}{5a^5}$ shall be less than an unit in the fifteenth place of decimals. If therefore we assume $a = 100{,}000$, we may take $n = 100$. And, since the last coefficient of q, viz. $\frac{n^3-n}{6} = 166650$, we must take the value of q to seven or eight places beyond the limit of the Table. Now,

$$q = \frac{4m}{a^3-3a} = \frac{4m}{a^3} = .\ 00000,00000,00001,73717,793, \&c.$$

$$\overline{|\frac{1}{a}} = \frac{2m}{a} + \frac{2m}{3a^3} = .\ 86858,89638,35456,565, \&c.$$

Therefore, $\overline{|\frac{1}{a}} + q = .\ 00000,86858,89640,09174,358, \&c.$

By the continual addition of these differences, we find the logarithms of $\frac{a+1}{a-1}, \frac{a+2}{a-2}, \frac{a+3}{a-3} \ldots\ldots\ldots \frac{a+100}{a-100}$.

200. Next, assuming $a'\ (= a + 1) = 100{,}001$, and n as before, we have,

$$\text{Log.}\frac{a+2}{a} = \left|\overline{\frac{1}{a'}}\right. \qquad \left|\begin{array}{l}\left|\overline{\frac{1}{a'}}\right. + q'. \\ \left|\overline{\frac{1}{a'}}\right. + 3q'. \\ \left|\overline{\frac{1}{a'}}\right. + 6q.\end{array}\right| \left|\begin{array}{l}2q'. \\ 3q'.\end{array}\right| q'.$$

$$\text{Log.}\frac{a+3}{a-1} = {}^{2}\left|\overline{\frac{1}{a'}}\right. + q'.$$

$$\text{Log.}\frac{a+4}{a-2} = {}^{3}\left|\overline{\frac{1}{a'}}\right. + 4q'.$$

$$\text{Log.}\frac{a+5}{a-3} = {}^{4}\left|\overline{\frac{1}{a'}}\right. + 10q'.$$

&c. &c.

$$\text{Here } q' = \frac{4m}{a'^{3}} = \begin{cases} \frac{4m}{a^{3}} \ldots .00000,\ 00000,\ 00001,\ 73717,\ 793, \text{ &c.} \\ -\frac{12m}{a^{4}}. \qquad\qquad 5,\ 211, \text{ &c.} \end{cases}$$

that is, $q' = .00000,\ 00000,\ 00001,\ 73712,\ 582, \text{ &c.}$

$$\left|\overline{\frac{1}{a'}}\right. = \begin{cases} \frac{2m}{a'} = .00000,\ 86858,\ 02780,\ 03723,\ 563, \text{ &c.} \\ +\frac{2m}{3a'^{3}} \left(=\frac{1}{6}q'\right) \qquad 28952,\ 097, \text{ &c.} \end{cases}$$

that is, $\left|\overline{\frac{1}{a'}}\right. = .00000,\ 86858,\ 02780,\ 32675,\ 660, \text{ &c.}$

Therefore, $\left|\overline{\frac{1}{a'}}\right. + q' = .00000,\ 86858,\ 02782,\ 06388,\ 242, \text{ &c.}$

By the continual addition of these differences, we get 99 logarithms, viz.:

those of $\frac{a+2}{a}, \frac{a+3}{a-1}, \frac{a+4}{a-2} \ldots\ldots\ldots \frac{a+100}{a-98}$.

Then, finding the log. $\frac{a+1}{a} = \frac{1}{2}\log. \frac{a+1}{a-1} - \frac{m}{2a^2} - \frac{m}{4a^4}$, the logarithms of 200 numbers from 99900 to 100100, may be obtained from two continued series of alternate addition and subtraction. Thus,

$$\text{L.}\, a + \text{l.}\, \frac{a+1}{a} - \text{l.}\, \frac{a+1}{a-1} + \text{l.}\, \frac{a+3}{a-1} - \text{l.}\, \frac{a+3}{a-3} + \text{l.}\, \frac{a+5}{a-3} - \&c.$$

$$\text{L.}\, a + \text{l.}\, \frac{a+2}{a} - \text{l.}\, \frac{a+2}{a-2} + \text{l.}\, \frac{a+4}{a-2} - \text{l.}\, \frac{a+4}{a-4} + \text{l.}\, \frac{a+6}{a-4} - \&c.$$

201. By similar processes, putting $a = 100200$, 100400, 100600, and 100800 successively, the Table of 1000 logarithms may be completed.

And in what manner the logarithms of the first thousand numbers may be obtained from these, will be obvious from the following Table, in which the numbers from 1009 down to 450 are divided into classes, and each class multiplied by such a factor as will produce numbers whose logarithms have been found.

TABLE.

Class 1.	1009 ....×100 999	4.	979 .....×103 970	7.	951 .....×106 943
2.	999 .....×101 990	5.	970 .....×104 961	8.	942 .....×107 934
3.	989 .....×102 980	6.	960 .....×105 952	9.	934 .....×108 925
	&c.		&c.		&c.

All the classes of this Table, (except a few at the beginning,) are so connected, that the last number of one class is the same with the first of the next following. So that we have only to show how the logarithm of one number in each of the few classes that are unconnected may be obtained. For, if the logarithm of one of the numbers, or of the multiplier, in any class be given, those of all the others will also be given.

In the thirteenth class, we have the log. 896 . 112 ($2^{11} \cdot 7^{2}$), from which we may find the log. 7, as the

log. 2 is supposed to be known; we have also the log. 900 . 112, from which we may find the log. 3. Then from log. 7, log. 3, and those of the numbers in the first and second classes, we may successively deduce the logarithms of 980, 972, 960, 945, and 936, which will serve to connect the several classes in which these numbers are found.

The logarithms of all numbers from 1 to 1009 being thus determined, the Table may be easily filled up to 100,000, using the rules of approximation [198.] for the primes. Thus, to find the log. 1013, (the first prime above 1009,) we have $6 \log. \frac{1013}{1005} = 6 \log. \frac{1012}{1006} - 14 \log. \frac{1011}{1007} + 14 \log. \frac{1010}{1008}$, where 1010, 1011, 1012, though above 1009, yet being composite numbers, have given logarithms.

THIRD METHOD.

202. MUCH fewer places of decimals than those we have assumed above, are sufficient for most of the practical purposes to which logarithms are applied. The following is an easy method of constructing a Table of logarithms, from 1 to 100,000, to eight places of decimals, the modulus and log. 2 being supposed given.

Put $a = 50{,}000$, and let it be required to find first the logarithms of the numbers from a to $a + 100$.

In the series $\frac{a+1}{a}, \frac{a+2}{a+1}, \frac{a+3}{a+2}, \frac{a+4}{a+3} \ldots\ldots \frac{a+100}{a+99}$, the sum of the numbers in any term being denoted by n, the logarithm of that term will be $\overline{\frac{1}{2a+n}}^{2m}$, or $\frac{2m}{2a+n}$, as the first term of the series is sufficient for our purpose. Let $2a=c$, then $\frac{2m}{c+n} = 2m\left(\frac{1}{c} - \frac{n}{c^2}\right)$. Hence the following

RULE.

1. Divide $2m$ (.86858, 89638, &c.) by c, and the quotient again by c.

2. Subtract the second quotient from the first for the first remainder.

3. Then, from the first remainder subtract the double of the second quotient continually, for the second, third, fourth, &c. remainder.

4. The log. a added to the first remainder gives the log. $a+1$, which, added to the second remainder, gives the log. $a + 2$, which, added to the third remainder, gives the log. $a + 3$, &c.

OPERATION.

1. Quot. =	.00000, 86858, 8964	
2. Quot. =	....., ..., 8686	
1. Rem. =	.00000, 86858, 0278	L. 50,000 = 4 . 69897, 00043, 4
2. Q×2 =	. 1, 7372	..... 86858, 0
2. Rem. =	.00000, 86856, 2906	L. 50,001 = 4 . 69897, 86901, 4
	1, 7372	..... 86856, 3
3. Rem. =	.00000, 86854, 5534	L. 50,002 = 4 . 69898, 73757, 7
	1, 7372	86854, 6
4. Rem. =	.00000, 86852, 8162	L. 50,003 = 4 . 69899, 60612, 3
&c.		86852, 8
		L. 50,004 = 4 . 69900, 47465, 1
		&c.

In like manner, putting $a = 50{,}100$; 50,200 ; 50,300, &c. or $c = 100{,}200$; 100,400 ; 100,600, &c. successively, the logarithms of all numbers up to 100,000 may be found, from which the logarithms of those under 50,000 may be deduced by a continual subtraction of the log. 2 from the logarithms of the even numbers.

OF LOGARITHMS TO TWENTY DECIMALS.

203. LET it be required to find the logarithm of a number (consisting of many figures) to twenty places of

decimals, by means of Callet's Table to the same number of decimals. Let the number expressed by the first eight figures of the given number be denoted by a, the square number that first exceeds a by s^2, and their difference by d: find a number r as near to the square root of d as may be, so that $s+r$ and $s-r$ may both be composed of tabular numbers. The product (p) of $s+r$ and $s-r$, or s^2-r^2, will therefore have a given logarithm. Take the difference between the given number and p; consider it as a decimal fraction ($=c$), and the preceding figures of p as integers. Divide the said fraction by p, and multiply the quotient (q) by the modulus m. The product mq must be added to the log. p, if a be greater than p, otherwise subtracted from it. The product $mq \cdot \frac{1}{2}q$, (found by a table of logarithms to seven places,) is next to be subtracted, and there will remain the logarithm required.

EXAMPLE 1.

To find the log. of 3.14159, 26535, 89793, 23846 to twenty places of decimals.

Here

$a = 31415926$

$s^2 = 31416025$ ..... $s = 5605$ } $s+r = 5615 = 5 \times 1123.$

$d = \overline{\quad\quad 99}$ ..... $r = \quad 10$ } $s-r = 5595 = 15 \times 373.$

3141592.65358,97932,3846

p=3141592.5)

c=.....15358, &c. 0.00000,00488,89151,99488 = q.

Log. p =	.49714,98514,61845,43707	$\frac{1}{2}q$ log.	$\bar{8}$.3881828
mq = +	212,32288,93630	mq log.	$\bar{8}$.3269968
$\frac{1}{2}mq^2$ = —	51901		$\bar{16}$.7151796
	.49714,98726,94133,85436		

The log. p = log. $(s+r)$ + log. $(s-r)$ is found by the table; and the product mq is most easily found by means of a table of the multipliers of m.

If p should not be found large enough to admit the second term $\frac{1}{2}mq^2$ to be found by logarithms, it may be increased by the means used in the following

EXAMPLE 2.

To find the log. of 0.43429,44819,03251,82765,1.

a = 43429448

s^2 = 43441281 .... s = 6591 $\left.\right\}$ $s+r$ = 6700

d = 11833 .... r = 109 $\left.\right\}$ $s-r$ = 6482 = 7 × 926.

434294,481903251, &c.

p = 434294,00 ..... log. .63778,38293,97017,59574,9

$\frac{48m}{43429424}$ $\left.\right\}$ (= A) + 04800,00267,36150,3

$+\frac{1}{12}A^3m^3$ + 4,9

p' = 43429448 ..... log. .63778,43093,97284,95730,1

$p' = 43429448$ log. $= .63778,43093,97284,95730,1$

$q = .00000,00043$ $mq = +$ 19,03251,83599,2

$82399,30576,8$ $\frac{1}{2}mq^2 = -$ 217,0

Log. required, .63778,43013,00536,78912,3

But the logarithm of the modulus may be more easily found thus:

Since $\log.(p+c) = \log. p + \log.\frac{p+c}{p}$.

$$= \text{l.}\, p + \frac{mc}{p} - \frac{1}{2}\cdot\frac{mc^2}{p^2} + \frac{1}{3}\cdot\frac{mc^3}{p^3} - \&c.$$

By substituting $(p+c)$ for m on one side of this equation, and m for $(p+c)$ on the other, we get,

$$\text{Log.}\, m = \log. p + c + \frac{c^2}{2p} - \frac{c^3}{6p^3}.$$

$m = .434294,48190,$ &c.

$p = .434294,$ ..... log. $= .63778,38293,97017,59574,9$

$c = 48190,$ &c. $+$ 04819,03251,82765,1

$\frac{c}{p} = .00000,11096,24475 \times \frac{c}{2} = +$ 267,36582,1

$-\frac{c^3}{6p^3} = -$ 9,8

Log. $m = .63778,43013,00536,78912,3$

In these examples, we find the fractional part only of the logarithm, the integral part being already known. The former is always affirmative; but the latter, (which

is termed the Index, because it marks the *place* of the first figure of the number, reckoning on either side from the place of units,) is affirmative or negative, according as the number is greater or less than unit. The complete logarithm in the last example is $\bar{1}.63778$, &c. and the logarithm of .00004,34294, &c. is $\bar{5}.63778$, &c. (4 being in the fifth place from that of units). Such logarithms, it is true, are, in effect, wholly negative, since the negative part exceeds the affirmative, and may be thus expressed, -0.36221, &c. -4.36221, &c.; but the former notation is preferable.

204. If a logarithm be given, and the corresponding number to 20 places be required, the best method of effecting it, is by means of a table constructed as follows: Let a table be formed of 1000 numbers to 22 decimal places, answering to all logarithms from .00000,001 to .00000,999, increasing by the first term, and of another 1000 numbers, answering to the logarithms from .00000,00000,1 to .00000,00099,9, also increasing by the first term. Both parts of the table may be constructed at once without much trouble; and the logarithms of other numbers are to be deduced from it, in this manner:

Find a number, whose logarithm may be had to 20 decimals, such that the first five figures of its logarithm

may be the same with those of the given logarithm. The number answering to the difference of these two logarithms is easily obtained from such a table as we have described, in the manner pointed out in the following

EXAMPLE.

To find the number (to 20 decimals) corresponding to this log. 0.49714,98726,94133,85435.

From a common table of logarithms we find the number $314157 = 667 \times 471$, having the first five figures of its logarithm the same with those of the given logarithm.

Given log.	0.49714,98726,94133,85435
Log. 3.14157, ...	0.49714,67410,45445,14025
Difference, ...	0.00000,31316,48688,71410

The number answering to this difference we find from the table thus:

LOGARITHMS.	NUMBERS.
. ....., 313	1.00000,72071,17312,21655,4
. ....., 00016, 4	37,76259,53964,0
. ....., ..., 0868	19986,43860,7
. ....., ..., 00008, 71	20,05551,6
. ....., ..., ..., 00410	944,1
Sum,	1.00000,72109,13558,27975,8

Sum,	1.00000,72109,13558,27975,8
Decimal of first number multiplied by the sum of the rest,	27,35999,0
Second decimal multiplied by the sum of those under it,	7,6
Number answering to the difference,	1.00000,72109,13585,63982,4
Which multiplied by 3.14157, gives	3.14159,26535,89793,23850,

for the number required.

The two products, 27359990 and 76, are found by a common table of logarithms, thus:

First decimal,	7207117	log. $\bar{6}$.8577615
Sum 2d, 3d, 4th dec.	0003796246	$\bar{9}$.5793544
First product,	00027,35&c.	log. $\overline{14}$.4371159

OF LOGARITHMIC SINES, TANGENTS, &c.

By means of the theorems for the construction of logarithms, we find the hyperbolic logarithms of sines, tangents, &c. in terms of the arc as follows, [138, 178.].

205. Log. sin. $(45+a) =$ log. sin. $45^\circ + a - a^2 + \frac{2}{3}a^3 - \frac{2}{3}a^4 + \frac{2}{3}a^5 - \frac{32}{45}a^6 + \frac{244}{315}a^7 -$ &c.

206. Log. cos. $(45+a) =$ log. sin. $45^\circ - a - a^2 - \frac{2}{3}a^3 - \frac{2}{3}a^4 - \frac{2}{3}a^5 -$ &c.

207. Hence log. tan. $(45+a) = 2a + \frac{4}{3}a^3 + \frac{4}{3}a^5 + \frac{488}{315}a^7 +$ &c.

208. Log. sec. $a = \frac{1}{2}a^2 + \frac{1}{12}a^4 + \frac{1}{45}a^6 + \frac{17}{2520}a^8 +$ &c. [145, 178.]

209. Log. ver. $(90+a) = a - \frac{1}{2}a^2 + \frac{1}{6}a^3 - \frac{1}{12}a^4 + \frac{1}{24}a^5 -$ &c. [142, 178.]

210. Log. tan. a

$=$ log. $a +$ log. $(1 + \frac{1}{3}a^2 + \frac{2}{15}a^4 + \frac{17}{315}a^6 +$ &c.) [144.]

$=$ log. $a + \frac{1}{3}a^2 + \frac{7}{2.5.9}a^4 + \frac{62}{5.7.9^2}a^6 +$ &c.) [178.]

211. Log. sin. $a =$ log. $a - \frac{1}{6}a^2 - \frac{1}{4.5.9}a^4 - \frac{1}{5.7.9^2}a^6 -$ &c. [208, 210.]

212. Log. tan. $a =$ log. $a + \frac{2}{3}$ log. sec. $a + \frac{1}{4.5}a^4 + \frac{4}{7.9^2}a^6 +$ &c.

213. Log. sin. $a =$ log. $a - \frac{1}{3}$ log. sec. $a + \frac{1}{4.5}a^4 +$ &c.

The same might also be found in terms of the natural

sines, tangents, &c. Thus, putting $s = \text{sin. } a$, $t = \text{tan. } a$, and $C = \text{cos. } 2a$.

Because tan. $(45 + a) = \frac{1+t}{1-t}$, and tan.^{2} $(45 + \frac{1}{2}a)$ $= \frac{1+s}{1-s}$, [63, 64.]

214. $\frac{1}{2}$ Log. tan. $(45+a) = t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \frac{1}{7}t^7 + \&c.$

215. Log. tan. $(45 + \frac{1}{2}a) = s + \frac{1}{3}s^3 + \frac{1}{5}s^5 + \frac{1}{7}s^7 + \&c.$

216. 2 Log. tan. $(45 + \frac{1}{2}a) = z - \frac{1}{2.3}z^3 + \frac{1.3}{2.4.5}z^5$ $- \frac{1.3.5}{2.4.6.7}z^7 + \&c.$ [185.], putting $z = \frac{2s}{1-s^2}$.

Also, because sec. $2a = \frac{1+t^2}{1-t^2}$, and tan.$^2 a = \frac{1+C}{1-C}$.

217. $\frac{1}{2}$ Log. sec. $2a = t^2 + \frac{1}{3}t^6 + \frac{1}{5}t^{10} + \frac{1}{7}t^{14} + \&c.$

218. Log. tan. $a = C + \frac{1}{3}C^3 + \frac{1}{5}C^5 + \frac{1}{7}C^7 + \&c.$

From the doctrine of logarithms, the following rules, (given by several writers,) for approximating any power or root of a given quantity, are easily deduced. Suppose a, b nearly equal, $s = a + b$, and $d = a - b$; then

219. Rule I. $\left|\frac{a}{b}\right|^n = \frac{s+nd}{s-nd}$.

220. RULE II. $a = b \cdot \frac{(n+1).a^n+(n-1).b^n}{(n-1).a^n+(n+1).b^n}$.

For, since hyp. $\log.\frac{a}{b} = 2\left(\frac{d}{s} + \frac{1}{3}.\frac{d^3}{s^3} + \&c.\right)$,

$n.\log.\frac{a}{b} = 2\left(\frac{nd}{s}+\frac{1}{3}.\frac{nd^3}{s^3}+\&c.\right) = 2\left(\frac{nd}{s}+\frac{1}{3}.\frac{n^3d^3}{s^3}+\&c.\right)$

nearly, and consequently the numbers (expressed in Rule 1.) corresponding to these logarithms are also nearly equal.

Again, the second rule is derived from the first, in this manner: Since

$a^n : b^n :: s + nd : s - nd$, $a^n + b^n : a^n - b^n :: s : nd$,

or, $n(a^n + b^n) : a^n - b^n :: s : d$.

Hence we have,

$n(a^n+b^n)+(a^n-b^n) : n(a^n+b^n)-(a^n-b^n) :: a : b$.

The two rules, though different in form, are in effect the same; and it appears that the greater s is in respect to d, and the less n differs from 1, (or the nearer both $\frac{a}{b}$ and n are to 1,) the nearer is the approximation.

SPHERICAL TRIGONOMETRY.

DEFINITIONS AND PRINCIPLES.

221. THE circumference of a semicircle, revolving about its diameter that remains fixed, describes the surface of a sphere.

222. If a sphere be cut by a plane in any direction, the figure of the section is a circle.

223. Circles of the sphere, whose planes pass through the centre, are called great circles; and all others, small circles.

224. The axis of a circle is the diameter of the sphere, which is perpendicular to the plane of that circle.

225. The poles of a circle are the extremities of its axis.

226. The arc of a great circle between the pole and circumference of another great circle is a quadrant.

227. Two great circles are perpendicular to each other, if the circumference of the one pass through a pole of the other; and conversely, if they be perpendicular to one another, they pass through each other's poles.

228. By a spherical angle or triangle, is meant that which is formed on the surface of a sphere by arcs of great circles.

229. A spherical angle is the same as the plane angle formed by tangents to its sides at the point of concourse, and is therefore the same as the inclination of the planes of these sides.

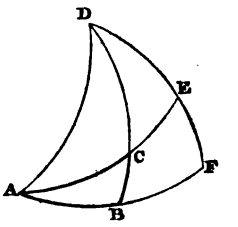
230. A spherical angle is measured by the arc of a great circle, whose pole is the angular point, intercepted between the sides of the angle.

231. Two great circles bisect each other; and the opposite angles at the same extremity, and at different extremities of the common diameter, are equal.

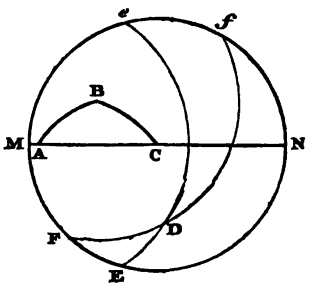
232. Each side of a spherical triangle is less than a semicircle; and any two sides, being produced, intersect again at the distance of a semicircle.

233. An arc of a great circle, joining two poles of two great circles, is the measure of the angle at their intersection.

234. If the hypotenuse AC and the two sides AB, BC, of the right-angled spherical triangle ABC, be produced to meet the great circle EF, whose pole is the angular point A, at E, F, D; the right-angled triangle CED formed thereby has its sides EC, CD, DE, and angle D, the complements of AC, CB, A and AB respectively, [226, 230]. With respect to ABC, therefore, CED is called the complemental triangle.



235. If the angular point C, of any spherical triangle ABC, be the pole of the great circle MEN, and the angular points A, B be the poles of the two semicircles ED*e*, FD*f*, which intersect in D, and meet the circumference of the former circle in E, *e*, and F, *f*; the surface of the hemisphere, on the former circle, will be divided into four triangles, one of which, viz. FD*e*, is termed the *Supplemental* Triangle, with respect to ABC. For the sides *e*D, DF, F*e*, of the triangle FD*e*, are the sup-



plements of the measures of the angles A, B, C opposite to them in the triangle ABC; and, *vice versâ*, the sides AB, BC, CA of the latter triangle are the supplements of the measures of the angles D, F, *e* opposite to them in the former.

Each of the other three triangles is termed a ***Reciprocal*** Triangle, with respect to ABC. Thus, the triangle DEF is the reciprocal of ABC; because the two sides of one are the measures of the angles at the base of the other, and the base of one measures the supplemental vertical angle of the other.

236. Two arcs, two angles, or an arc and an angle are said to be of *unlike* magnitude, (or simply, unlike,) if the one be greater and the other less than 90°; and of *like* magnitude, if they be both greater or both less.

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237. In a spherical triangle, equal sides subtend equal angles; the greater side subtends the greater angle; and conversely.

238. Any two sides are together greater than the third; and the sum of the three sides is less than a circle.

239. Any two angles are together greater than the

supplement of the third; and the sum of the three angles is greater than two right angles.

240. If the three sides, or three angles, be of like magnitude, the sides and angles are all of like magnitude.

241. Each of the two sides of a right-angled triangle is of like magnitude with its opposite angle, but less than the measure of that angle, if the angle be acute, and greater, if it be obtuse.

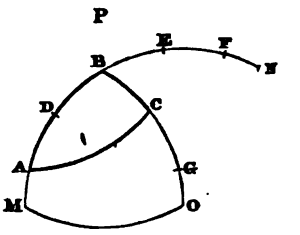
242. According as the two sides of a right-angled triangle are like or unlike, the hypotenuse is less or greater than a quadrant; and conversely.

243. Half the sum of two sides of any spherical triangle is of like magnitude with half the sum of the two opposite angles.

TRIGONOMETRICAL THEOREMS.

244. In any spherical triangle, the excess of the versed sine of the sum of the two sides above the versed sine of their difference, is to the excess of the versed sine of the base above the same, as the double of the radius is to the versed sine of the vertical angle.

Let ABC be a spherical triangle, having the side AB greater than BC. In BA, BC produced (if necessary) below the base AC, take the quadrants BM, BO; and in AB, continued beyond the vertex, take the quadrant BN. Also, make the arcs BD, BE each equal to BC, and BF, BG each equal to BA. Let the straight line which joins two points, as A, E, be expressed by $\overline{AE}$, the arc between them being denoted by AE.



Let $\overline{MD}$ meet the common diameter of the circles BM, BO in the point P; $\overline{OC}$ will meet it in the same point. For, let the circle BO revolve about the common diameter which remains fixed, (the point P, therefore, continuing fixed,) till it coincide with the equal circle BM; then the point C will fall upon D, because BC = BD, and the point O upon M, because BO = BM; consequently $\overline{OC}$ coincides with $\overline{MD}$, and passes through the point P. Hence the rectilineal triangles PMO, PDC are isosceles, and $\overline{DC}$ parallel to $\overline{MO}$. In like manner, $\overline{AG}$ is parallel to MO.

By similar reasoning, $\overline{EC}$, $\overline{FG}$, $\overline{NO}$ may be shewn to be parallel to one another.

But $\overline{DE}$, $\overline{AF}$, $\overline{MN}$, which intercept equal arcs in the same circle, are also parallel to one another.

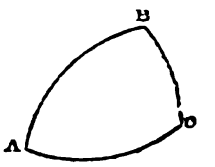
Therefore the rectilineal triangles DEC, AFG, MNO are similar, and $\overline{AF} \cdot \overline{DE} : \overline{AG} \cdot \overline{DC} : : \overline{MN}^2 : \overline{MO}^2 : :$ (because $\overline{MN}$ is a diameter,) $\overline{MN}$: ver. MO [8.] : : $\overline{MN}$: ver. B. [230.]

Now, from the properties of the circle, we have $\overline{AF} \cdot \overline{DE} = \overline{AE}^2 - \overline{AD}^2 = \overline{MN}$.(ver. AE—ver. AD) [8.]; also, (for it is plain that the circumference of a circle may pass through the four points A, D, C, G,) $\overline{AG} \cdot \overline{DC} = \overline{AC}^2 - \overline{AD}^2 = \overline{MN}$.(ver. AC—ver. AD.)

Consequently,

Ver. AE—ver. AD : ver. AC—ver. AD : : 2R : ver. B.

Thus, putting the sum of the two sides AB, BC $= S$, their difference $= D$, and half the sum of the three sides $= P$: Also, the sum of the angles A, C $= s$, their difference $= d$, and half the sum of the three angles $= p$; we have,



Ver. S — ver. D : 2R : : ver. AC — ver. D : ver. b. [244.]

245. Ver. S–ver. D : 2R : : ver. S–ver. AC : suvers. B. [244, 2.]

[illegible] Cos. D−Cos. S : 2R : : 2 cos. AC−cos. S−cos. [illegible] : : cos. B. [244, 245, 2.]

[illegible] Sin. AB . sin. BC : R² : : cos. AC − [illegible] : [illegible] − ½ cos. D : cos. B. [246, 8.]

[illegible] Sin. AB . sin. BC : R² : : sin.² ½ AC−sin.² [illegible] : sin.² ½ B. [244, 8.]

[illegible] Sin. AB . sin. BC : R² : : sin. [illegible] − ½ [illegible] . sin. ½ AC − ½ D) : sin.² ½ B. [245, 9.]

[illegible] Sin. AB . sin. BC : R² : : sin. [illegible] P − BC] . sin. [illegible] P − AB] : sin.² ½ B. [245.]

[illegible] Sin. AB . sin. BC : R² : : sin.² ½ S − sin.² ½ AC : cos.² ½ B. [245, 8, 9.]

252. Sin. AB . sin. BC : R² : : sin. (½ S + ½ AC) . sin. (½ S − ½ AC) : cos.² ½ B. [245, 46.]

253. Sin. AB . sin. BC : R² : : sin. P . sin. (P−AC) : cos.² ½ B. [252.]

254. Sin. P . sin. (P−AC) : sin. (P−BC) . sin. (P−AB) : : R² : tan.² ½ B. [253, 250, 12.]

255. Tan. ½A : tan. ½C : : sin. (P−AB) : sin. (P−BC) : : sin. (½AC − ½D) : sin. (½AC + ½D). [254.]

256. Cot. ½ A : tan. ½ C : : sin. P . sin. (P − AC) : : sin. (½S+½AC) : sin. (½S−½AC). [254, 255.]

257. Sin. ½ *s* : sin. ½ *d* : : tan. ½ AC : tan. ½ D. [255, 47, 43.]

258. Cos. $\frac{1}{2}s$: cos. $\frac{1}{2}d$: : tan. $\frac{1}{2}$AC : tan. $\frac{1}{2}$S. [256, 48, 13.]

259. Tan. $\frac{1}{2}s$: tan. $\frac{1}{2}d$: : tan. $\frac{1}{2}$S : tan. $\frac{1}{2}$D. [257, 258, 12.]

260. Sin. A : sin. C : : sin. BC : sin. AB. [259, 47.]

261. Sin. AB . sin. BC : $2\sqrt{(\text{sin.}^2 \frac{1}{2}\text{AC} - \text{sin.}^2 \frac{1}{2}\text{D})}$. $\sqrt{(\text{sin.}^2 \frac{1}{2}\text{S} - \text{sin.}^2 \frac{1}{2}\text{AC})}$: : R : sin. A. [248, 251, 12.]

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262. 2R . ver. AC = ver. B . ver. S + suvers. B . ver. D. [244.]

263. 2R . suvers. AC = ver. B . suvers. S + suvers. B . suvers. D. [245.]

264. 2R . cos. AC = ver. B . cos. S + suvers. B . cos. D. [262, 263.]

265. 2cos. AC = cos. S + cos. D + $\frac{1}{2}$cos. (B + D) + $\frac{1}{2}$cos. (B ~ D) — $\frac{1}{2}$cos. (S + B) — $\frac{1}{2}$cos. (S ~ B). [246, 27.]

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266. Cos. B . sin. AB . sin. BC + R . cos. AB . cos. BC = R^2 . cos. AC. [257, 27.]

267. Cos. AC . cosec. AB . cosec. BC — R . cot. AB . cot. BC = R^2 . cos. B. [266, 4.]

268. R . cos. A . sin. AC + cos. B . sin. BC . cos. AB = R . sin. AB . cos. BC. [266, 3.]

269. Cot. A . sin. B . cosec. AB + R . cos. B . cot. AB = R^2 . cot. BC. [268, 260.]

270. Cot. BC . sin. AB . cosec. B − R . cos. AB . cot. B = R^2 . cot. A. [269, 4, 6.]

These theorems being applied to the reciprocal triangle FDE, [235.] and the analogies and equations so obtained expressed in terms of the triangle ABC, we have the similar theorems:

271. Ver. s — ver. d : 2R : : suvers. B — ver. d : suvers. AC.

272. Ver. s — ver. d : 2R : : ver. s — suvers. B : ver. AC.

273. Cos. d — cos. s : 2R : : 2cos. B + cos. s + cos. d : 2cos. AC.

274. Sin. A . sin. C : R^2 : : cos. B + $\frac{1}{2}$cos. s + $\frac{1}{2}$cos. d : cos. AC.

275. Sin. A . sin. C : R^2 : : cos.^{2} $\frac{1}{2}$ B — sin.^{2} $\frac{1}{2}$ d : cos.^{2} $\frac{1}{2}$ AC.

276. Sin. A . sin. C : R^2 : : cos ($\frac{1}{2}$ B + $\frac{1}{2}$ d) . cos ($\frac{1}{2}$ B − $\frac{1}{2}$ d) : cos.^{2} $\frac{1}{2}$ AC.

277. Sin. A . sin. C : R^2 : : cos. (p—A) . cos. (p—C) : cos.^{2} $\frac{1}{2}$ AC.

278. Sin. A . sin. C : R^2 : : sin.^{2} $\frac{1}{2}s$ — cos.^{2} $\frac{1}{2}$ B : sin.^{2} $\frac{1}{2}$ AC.

279. Sin. A . sin. C : R^2 : : cos. ($\frac{1}{2}s$ + $\frac{1}{2}$ B) . cos ($\frac{1}{2}s$ — $\frac{1}{2}$ B) : sin.^{2} $\frac{1}{2}$ AC.

280. Sin. A . sin. C : R^2 : : cos. p . cos. (p — B) : sin.^{2} $\frac{1}{2}$ AC.

281. Cos (p—A) . cos (p—C) : cos p . cos (p—B) : : R^2 : tan.^{2} $\frac{1}{2}$ AC.

282. Tan. $\frac{1}{2}$AB : tan. $\frac{1}{2}$BC : : cos (p–C) : cos (p–A) : : cos ($\frac{1}{2}$B — $\frac{1}{2}d$) : cos ($\frac{1}{2}$B + $\frac{1}{2}d$).

283. Cot. $\frac{1}{2}$ AB : tan. $\frac{1}{2}$ BC : : cos (p — B) : cos p : : cos ($\frac{1}{2}$ S — $\frac{1}{2}$ B) : cos ($\frac{1}{2}$ S + $\frac{1}{2}$B).

284. Sin. $\frac{1}{2}$ S : sin. $\frac{1}{2}$ D : : cot. $\frac{1}{2}$ B : tan. $\frac{1}{2}d$.

285. Cos. $\frac{1}{2}$ S : cos. $\frac{1}{2}$ D : : cot. $\frac{1}{2}$ B : tan. $\frac{1}{2}s$.

286. Cos. AC . sin. A . sin. C — R . cos. A . cos. C = R^2 . cos. B.

287. Cos. B . cosec. A . cosec. C + R . cot. A . cot. C = R^2 . cos. AC.

288. 2R.suvers. B=suvers. AC.ver. s+ver. AC.ver. d.

289. 2R.ver. B=suvers. AC.suvers. s+ver. AC.suvers. d.

290. 2R.cos. B=suvers. AC.cos. s+ver. AC.cos. d.

291. 2 Cos. B = cos. s + cos. d + $\frac{1}{2}$ cos. (AC + s) + $\frac{1}{2}$cos. (AC ~ s) – $\frac{1}{2}$cos. (AC + d) – $\frac{1}{2}$cos. (AC ~ d).

From the comparison of these with the former, several other useful theorems may be deduced. Thus, putting

$$\text{Sin.}^2 M = \text{sin.}^2 \tfrac{1}{2}AC - \text{sin.}^2 \tfrac{1}{2}D, \text{ and}$$
$$\text{Sin.}^2 N = \text{cos.}^2 \tfrac{1}{2}B - \text{sin.}^2 \tfrac{1}{2}d.$$

Since sin. AB . sin. BC : R^2 : : $\text{sin.}^2 M$: $\text{sin.}^2 \frac{1}{2}B$, [248.]
and R^2 : sin. A . sin. C : : $\text{cos.}^2 \frac{1}{2}AC$: $\text{sin.}^2 N$. [275.]
Therefore,

$$\text{Sin.}^2 AC : \text{sin.}^2 B : : \text{cos.}^2 \tfrac{1}{2}AC : \text{sin.}^2 \tfrac{1}{2}B + \text{sin.}^2 M : \text{sin.}^2 N. \quad [260.]$$

$$\text{Sin.}^2 \tfrac{1}{2}AC : \text{cos.}^2 \tfrac{1}{2}B : : \text{sin.}^2 M : \text{sin.}^2 N. \quad [12.]$$

And subtracting the terms of the latter ratio from those of the former,

$$\text{Sin.}^2 \tfrac{1}{2}D : \text{sin.}^2 \tfrac{1}{2}d : : \text{sin.}^2 \tfrac{1}{2}AC : \text{cos.}^2 \tfrac{1}{2}B.$$

Hence

292. Sin. $\frac{1}{2}d$: sin. $\frac{1}{2}D$: : cos. $\frac{1}{2}B$: sin. $\frac{1}{2}AC$.

293. Sin. $\frac{1}{2}s$: cos. $\frac{1}{2}D$: : cos. $\frac{1}{2}B$: cos. $\frac{1}{2}AC$. [292, 257.]

294. Cos. $\frac{1}{2}d$: sin. $\frac{1}{2}S$: : sin. $\frac{1}{2}B$: sin. $\frac{1}{2}AC$. [292, 284.]

295. Cos. $\frac{1}{2}s$: cos. $\frac{1}{2}S$: : sin. $\frac{1}{2}B$: cos. $\frac{1}{2}AC$. [293, 285.]

296. Suppose tan. $\frac{1}{2}$S : tan. $\frac{1}{2}$AC : : R : tan. x,
and tan. $\frac{1}{2}$AC : tan. $\frac{1}{2}$D : : R : tan. y.

Then, [47, 54.]

Sin ($\frac{1}{2}$S+$\frac{1}{2}$AC) : sin ($\frac{1}{2}$S–$\frac{1}{2}$AC) : : R : cot (45+x)
: : cot $\frac{1}{2}$A : tan $\frac{1}{2}$C. [256.]

and

Sin ($\frac{1}{2}$AC+$\frac{1}{2}$D) : sin ($\frac{1}{2}$AC–$\frac{1}{2}$D) : : R : cot (45+y)
: : tan. $\frac{1}{2}$C : tan. $\frac{1}{2}$A.

Consequently tan.^{2} $\frac{1}{2}$A = cot. (45+x) . cot. (45+y).

297. Suppose

(1.) Sin. $\frac{1}{2}$AC : sin. $\frac{1}{2}$D (: : cos. $\frac{1}{2}$B : sin. $\frac{1}{2}d$) : : R : sin. m. [292.]

(2.) Sin. $\frac{1}{2}$S : sin. $\frac{1}{2}$AC (: : cos. $\frac{1}{2}d$: sin. $\frac{1}{2}$B) : : R : sin. n. [294.]

(3.) Cos. $\frac{1}{2}$D : cos. $\frac{1}{2}$AC (: : sin. $\frac{1}{2}s$: cos. $\frac{1}{2}$B) : : R : sin. q. [293.]

(4.) Cos. $\frac{1}{2}$AC : cos. $\frac{1}{2}$S (: : sin. $\frac{1}{2}$B : cos. $\frac{1}{2}s$) : : R : sin. r. [295.]

Then (1.) R : tan. $\frac{1}{2}$B : : cot. n : cos. m.
(2.) R : cot. $\frac{1}{2}$B : : cot. q : cos. r.
(3.) R : tan. $\frac{1}{2}$AC : : cot. n : cos. r.
(4.) R : cot. $\frac{1}{2}$AC : : cot. q : cos. m.
(5.) R : tan. $\frac{1}{2}$D : : cot. m : cos. q.
(6.) R : tan. $\frac{1}{2}d$: : cot. m : cos. n.
(7.) R : cot. $\frac{1}{2}$S : : cot. r : cos. n.
(8.) R : cot. $\frac{1}{2}s$: : cot. r : cos. q.

298. Also,

(1.) Sin. AB : cos. m : : cos. r : sin. A. [261.]

(2.) Sin. AB : cot. n : : cot. q : sin. A.

299. And,

(1.) Sin. $\frac{1}{2}$D : sin. $\frac{1}{2}$B : : $\sqrt{}$(sin. AB . sin. BC) : cot. m.

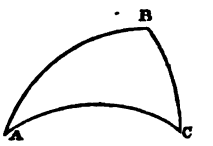
(2.) Sin. $\frac{1}{2}$S : cos. $\frac{1}{2}$B : : $\sqrt{}$(sin. AB . sin. BC) : cos. n.

(3.) Cos. $\frac{1}{2}$D : sin. $\frac{1}{2}$B : : $\sqrt{}$(sin. AB . sin. BC) : cos. q.

(4.) Cos. $\frac{1}{2}$S : cos. $\frac{1}{2}$B : : $\sqrt{}$(sin. AB . sin. BC) : cot. r.

RIGHT-ANGLED SPHERICAL TRIANGLES.

By supposing the angle B to be a right angle, we get the following theorems, by which all the cases of a right-angled triangle may be resolved.



300. R . cos. AC = cot. A . cot. C = cos. AB . cos. BC. [287, 266.]

301. R . sin. BC = tan. AB . cot. C = sin. AC . sin. A. [300, 234.]

302. R . cos. A = tan. AB . cot. AC = cos. BC . sin. C. [301, 234.]

NAPIER'S PRACTICAL RULE.

THESE theorems were reduced by Napier, the illustrious inventor of logarithms, into the following practical rule, which may be more easily remembered:

The hypotenuse, the two oblique angles, and the *complements* of the two sides, are the five circular parts of a right-angled triangle, any one of which being considered as the middle part, and denoted by M, the two that are adjacent to it being denoted by A, a; and the remaining two, disjoined from it, by D, d; then

303. R . cos. M = cot. A . cot. a = sin. D . sin. d.

[300, 301, 302.]

When two parts are given, to find the other three, each of the given parts must be made the middle part (successively), to find two of the required parts; and the third required part must be the middle part to find itself.

QUADRANTAL TRIANGLES.

304. The same rule will also serve for the cases of a quadrantal triangle, (that is, a triangle of which (only)

one of the sides is a quadrant,) as it is the reciprocal of a right-angled triangle, [235.]. Its circular parts are, the angle opposite to the quadrant, the including sides, and the *complements* of the two remaining angles.

CASES OF OBLIQUE-ANGLED TRIANGLES.

305. Of the three sides and three angles of a spherical triangle, any three being given, the other three may be found by means of the preceding theorems. There are six cases of this general problem, viz.

CASE I.—When two sides and the included angle are given.

Find the two remaining angles by 284, 285.

Find one of them separately by 270.

Find the remaining side by 284 and 292, or 285 and 293, &c.

CASE II.—When the three sides are given.

Find an angle by 249, or 252, or 254, or 296, or 297, &c.

Find all the angles by 254 and 256, or 296.

CASE III.—When two sides and an angle opposite to one of them are given.

Find the angle opposite to the other by 260.

Find the remaining angle by 284 or 285, and the remaining side by 257 or 258.

The other three cases, when two angles and the included side,—the three angles,—two angles and a side opposite to one of them, are given, may be solved by the corresponding theorems; but it seems better to reduce them, by means of the reciprocal triangle, to the preceding cases.

THEOREMS RESPECTING RIGHT-ANGLED TRIANGLES.

A GREAT variety of theorems respecting right-angled triangles may be easily deduced from those already delivered [300, 301, 303]: Some of the most useful are subjoined.

306. $2\cos AC = \cos(AB+BC)+\cos(AB-BC)$.
[300, 27.]

307. $2\sin BC = \cos(AC-A)-\cos(AC+A)$.
[301, 28.]

308. $2\cos A = \sin(BC+C)+\sin(BC-C)$.
[302, 25.]

These are adapted to the use of the natural sines, the radius being supposed 1.

309. $R^2 : \tan^2 \frac{1}{2}AC :: \cos(A - C) : -\cos(A+C)$.
[300, 10, 48.]

310. $R^2 : \tan^2(45+\frac{1}{2}BC) :: \sin(C-AB) : \sin(C+AB)$.
[301, 10, 47.]

311. $R^2 : \tan^2 \frac{1}{2}A :: \sin(AC+AB) : \sin(AC-AB)$.
[302, 10, 47.]

It appears from the first of these analogies, that $(A + C)$ must *always* be greater than 90°. This is all that is indicated by the sign — prefixed to the last term: And, this being well known from other principles, the sign might have been omitted. It is retained, to shew that a due regard paid to the signs, leads to the knowledge of essential truths.

312. $R^2 : \tan^2 \frac{1}{2}AB :: \cot(\frac{1}{2}AC+\frac{1}{2}BC) : \tan(\frac{1}{2}AC-\frac{1}{2}BC)$.
[300, 10, 44.]

313. $R^2 : \cot^2(45+\frac{1}{2}A) :: \tan(\frac{1}{2}AC+\frac{1}{2}BC) : \tan(\frac{1}{2}AC-\frac{1}{2}BC)$.
[301, 10, 43.]

314. $R^2 : \cot^2(45+\frac{1}{2}AC) :: \tan(\frac{1}{2}A+\frac{1}{2}BC) : \tan(\frac{1}{2}A-\frac{1}{2}BC)$.
[301, 10, 43.]

315. $R^2 : \cot^2(45+\frac{1}{2}C) :: \cot(\frac{1}{2}A+\frac{1}{2}BC) : \tan(\frac{1}{2}A-\frac{1}{2}BC)$.
[302, 44.]

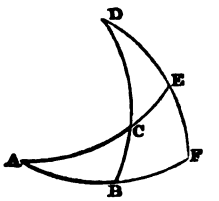
316. $R^2 : \tan^2 \frac{1}{2}BC :: \tan(\frac{1}{2}C+\frac{1}{2}{}^cA) : \tan(\frac{1}{2}C-\frac{1}{2}{}^cA)$.
[302, 43.]

where ${}^cA$ denotes the complement of A.

Putting AB+BC = S, AB—BC = D, C+A = s, C — A = d.

317. Sin.$\frac{1}{2}$S : sin.$\frac{1}{2}$D : : R : tan.$\frac{1}{2}d$. [284.]

318. Cos.$\frac{1}{2}$S : cos.$\frac{1}{2}$D : : R : tan.$\frac{1}{2}s$. [285.]



319. Tan. $\frac{1}{2}$BC : tan. $\frac{1}{2}$AB : : R : tan. (45 + $\frac{1}{2}d$). [282.]

320. Tan. $\frac{1}{2}$BC : cot $\frac{1}{2}$AB : : R : tan (45+$\frac{1}{2}s$). [283.]

321. Sin. 2A : sin. 2C : : cos. BC : cos. AB. [300, 12.]

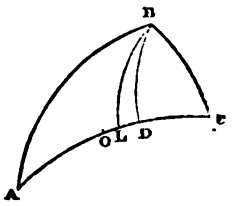
322. Sin. 2AB : sin. 2C : : sin. AC : sin. A. [321, 234.]

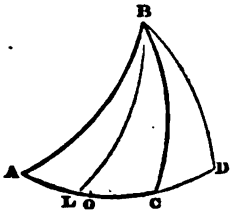
323. Sin. 2AC : sin. 2AB : : cos. BC : sin. C. [322, 234.]

TWO RIGHT-ANGLED TRIANGLES HAVING ONE SIDE COMMON.

In any spherical triangle ABC, let the perpendicular BD be drawn from the vertex B to the base AC, (produced if necessary,) and there will be formed two right-angled triangles, ADB, CDB, having the side BD common to both. Respecting such triangles, we

have the following theorems, (the vertical angle being bisected by BL, and the base AC bisected in O).





324. Cos. AD : cos. DC : : cos. AB : cos. BC. [300.]

325. Sin. AD : sin. DC : : tan. C : tan. A. [301.]

326. Tan. AD : tan. DC : : tan. ABD : tan. CBD. [301.]

327. Cos. ABD : cos. CBD : : cot. AB : cot. BC. [302.]

328. Sin. ABD : sin. CBD : : cos. A : cos. C. [302.]

329. Tan. $\frac{1}{2}$AC : tan. $\frac{1}{2}$S : : tan. $\frac{1}{2}$D : tan. DO. [324, 44.]

330. Tan. $\frac{1}{2}$AC : tan. DO : : sin. s : sin. d. [325.]

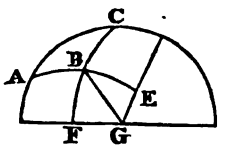
331. Sin. AC : sin. 2DO : : sin. ABC : sin. 2DBL. [326.]

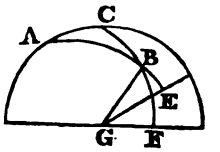
332. Cot. $\frac{1}{2}$ABC : tan. DBL : : sin. S : sin. D. [327.]

333. Tan. $\frac{1}{2}$ABC : tan. DBL : : cot. $\frac{1}{2}s$: tan. $\frac{1}{2}d$. [328.]

TWO RIGHT-ANGLED TRIANGLES HAVING A COMMON HYPOTENUSE.

LET the extremities of the base AC, in any spherical triangle ABC, be the poles of two great circles EG, GF intersecting in G, and meeting the sides AB, BC (produced if necessary) in E, F, and let the arc BG be drawn: BEG, BFG are two right-angled triangles having the same hypotenuse BG. Respecting such triangles, we have the following theorems:





334. Cos. BE : cos. BF : : cos. GF : cos. GE. [300.]

335. Tan. GBE : tan. GBF : : tan. BGF : tan. BGE. [300.]

336. Sin. GE : sin. GF : : sin. GBE : sin. GBF. [301.]

337. Tan. GE : tan. GF : : cos. BGE : cos. BGF. [302.]

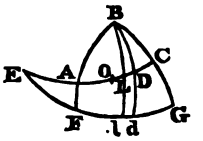
338. Sin. GE : sin. BF : : sin. GBE : sin. BGF. [301.]

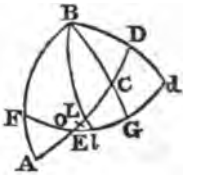
339. Tan. GE : tan. BF : : cos. BGE : cos. GBF. [302.]

Other theorems may be obtained as in the preceding case. Or, since the arcs GE, GF are the complements of the measures of the angles at A and C, GB the complement of the perpendicular upon AC, &c. they may be obtained by only expressing the theorems respecting ABC, in terms of the triangles BEG, BFG.

TWO RIGHT-ANGLED TRIANGLES HAVING ONE ANGLE COMMON.

Let the vertex B, of any spherical triangle ABC, be the pole of the great circle EFG, meeting the base AC (produced if necessary) in E, and the sides AB, BC in F, G: Then EFA, EGC are two right-angled triangles, having the same angle (or equal angles) at the point E. Respecting such triangles, we have the following theorems:





340. Cos. EA : cos. EC : : tan. C : tan. A.

341. Sin. EA : sin. EC : : sin. AF : sin. CG.

342. Tan EA : tan EC : : tan EF : tan EG.

343. Sin EF : sin EG : : tan AF : tan CG.

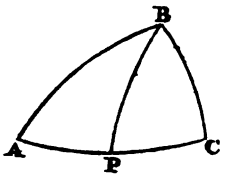
344. Cos EF : cos EG : : cos A : cos C.

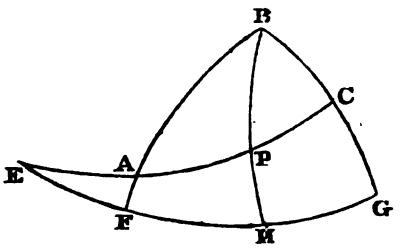
345. Sin A : sin C : : cos CG : cos AF.

Here, in like manner, the theorems respecting ABC may be expressed in terms of the triangles EAF, ECG.

FOUR GENERAL THEOREMS,

Respecting any spherical triangle having an arc drawn from the vertex to any point in the base.





346. Cot P . sin AC = cot A . sin PC — cot C . sin AP.

347. Cos BP . sin AC = cos AB . sin PC + cos BC . sin AP.

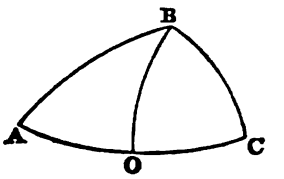
348. Cot BP . sin B = cot AB . sin CBP + cot BC . sin ABP.

349. Cos P . sin B = cos A . sin CBP — cos C . sin ABP.

Which may be thus demonstrated: Let B be the pole of EG, meeting CA, BA, BP, BC in E, F, H, G; then, since cos EP . sin AC = cos EC . sin AP

+cos EA . sin PC, [73.] and cot P, cot C, cot A are proportional to cos EP, cos EC, cos EA, [340.] therefore cot P . sin AC = cot C . sin AP + cot A . sin PC. But C in the triangle ABC being supposed acute, C of the triangle ECG must be obtuse, and therefore its cotangent negative, &c.

Hence, if the arc BO bisect the base,



350. Sin AB : sin BC : : sin CBO : sin. ABO.

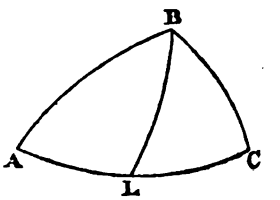
351. Tan $\frac{1}{2}$S : tan $\frac{1}{2}$D : : tan $\frac{1}{2}$B : tan $\frac{1}{2}$ (CBO—ABO).

352. Cos BO . cos $\frac{1}{2}$AC = cos $\frac{1}{2}$S . cos $\frac{1}{2}$D. [347.]

353. 2Cot O . cos $\frac{1}{2}$AC = cot A—cot C = $\frac{\sin d}{\sin A \cdot \sin C}$ [346.]

354. Cos O . sin BO . sin $\frac{1}{2}$AC = R . sin $\frac{1}{2}$S . sin $\frac{1}{2}$D. [266.]

And, if the arc BL bisect the vertical angle,



355. Sin AB : sin BC : : sin AL : LC.

356. Tan $\frac{1}{2}$S : tan $\frac{1}{2}$D : : tan $\frac{1}{2}$AC : tan $\frac{1}{2}$ (AL—LC).

357. Cos L . Cos $\frac{1}{2}$B = sin $\frac{1}{2}s$. sin $\frac{1}{2}d$. [349.

358. 2Cot BL . cos $\frac{1}{2}$B = $\frac{R^2 . \text{sin. S}}{\text{sin AB . sin BC}}$. [348.]

359. Cos L . sin AC = sin D . cos $\frac{1}{2}$B. [357.]

DISTINCTION OF CASES.

In the preceding theorems we have supposed certain arcs and angles less than 90°; it will now be proper to take notice of those cases in which such arcs or angles are greater than 90°, and to shew what changes take place in the expression of the theorems.

360. Analogies, whose terms are simple, admit of no change in the expression: And, with respect to the signs of the terms, we have only to observe that they are affirmative or negative in pairs, so that the fourth term is negative, if all or only one of the preceding terms be negative. Thus, in Theorem 305,

$$\cos \tfrac{1}{2}S : \cos \tfrac{1}{2}D : : \cot \tfrac{1}{2}B : \tan \tfrac{1}{2}s,$$

as an angle and a side of a spherical triangle are each less than 180°, $\frac{1}{2}$B and $\frac{1}{2}$D must be each less than 90°: the second and third terms therefore are always affirmative; consequently, the fourth term is affirmative or negative, according as the first term is affirmative or

negative. Hence, ½sum of two sides, and ½sum of the two opposite angles, are of like magnitude, [236.]

361. Analogies or equations which are composed entirely of versed sines, [264, 265, 282, &c.] remain the same in all cases, because the versed sines of all arcs from 0 to 360° are affirmative.

362. All other analogies and equations, in which are found cosines, tangents, &c. may be resolved each into four cases. These we shall exemplify in a few instances.

In the Theorem [266.] by which the vertical angle may be found from the base, and the sum and difference of the sides, we have supposed S (and consequently D and AC) less than 90°; the theorem being expressed thus:

Case I. Cos D — cos S : R : : 2cos AC — cos S — cos D : cos B.

And, if we suppose S greater than 270°, as its cosine is then affirmative, the expression of the theorem will still be the same.

But, if S be between 90° and 270°, its cosine is negative, and the theorem becomes,

Case II. Cos D + cos S : R : : 2 cos AC + cos S — cos D : cos B.

And, if at the same time AC be greater than 90°, it is further changed to,

CASE III. Cos D + cos S : R : : cos S — cos D — 2 cos AC : cos B.

Lastly, if both S and D (and consequently AC) be greater than 90°, it will be thus expressed:

CASE IV. Cos S — cos D : R : : cos S + cos D — cos AC : cos B.

In each case, the cos B is negative (or the angle B obtuse,) when the predominant signs of the first and third terms are unlike.

363. Of the Theorem [286.] by which the base AC may be found when the two sides AB, BC, and the included angle B are given, the several cases are as follows:

CASE I. When AB, BC are like, and B acute.

$$\text{Cos B . sin AB . sin BC} + \text{R . cos AB . cos BC} = \text{R}^2 \text{ . cos AC.}$$

For, when AB, BC are both greater than 90°, as two negative cosines give an affirmative product, there is no change of signs. In this case AC is less than 90°.

CASE II.—When AB, BC are unlike, and B obtuse.

Here the first and second terms of the equation become both negative, consequently the third term must also be negative. And, as an equation is not affected by changing all its signs, the expression of the theorem is still the same.

CASE III. When AB, BC are like, and B obtuse.

$$-\cos B \cdot \sin AB \cdot \sin BC + R \cdot \cos AB \cdot \cos BC = R^2 \cdot \cos AC.$$

Whence it appears that AC is greater or less than 90°, according as the first term of the equation is greater or less than the second.

CASE IV. When AB, BC are unlike, and B acute.

$$\text{Cos } B \cdot \sin AB \cdot \sin BC - R \cdot \cos AB \cdot \cos BC = R^2 \cdot \cos AC.$$

In this case, AC is greater or less than 90°, according as the second term is greater or less than the first.

364. The theorem we have been considering, and others similarly expressed, may be reduced to a more convenient form for logarithms; thus,

$$R : \cos B :: \tan AB \cdot \tan BC : \tan^2 x$$

CASES I. II. $R^2 : \sec^2 x$ } $:: \cos AB \cdot \cos BC$

III. IV. $R \cdot \sec 2x : \sec^2 x$ } $: R \cdot \cos AC$

Or thus,

$$R : \sec B :: \cot AB : \tan z$$ } if B, AB unlike

$$\text{Sin } z : \sin (z + BC) :: \cos AB : \cos AC$$ } $z > 90°$.

365. Theorems [286, 287.] may be considered as different forms of one proposition; Theorems [289, 290.] as similar forms of another; and Theorems [306, 307.] as similar forms of a third.

FIRST FORM.	SECOND FORM.
CASE I. $+a+b=x$	CASE I. $+a-b=x$
CASE II. $-a-b=x$	CASE II. $-a+b=x$
CASE III. $-a+b=x$	CASE III. $-a-b=x$
CASE IV. $+a-b=x$	CASE IV. $+a+b=x$

In the first and second cases of the first form, and in the third and fourth of the second form, the part required is greater or less than 90°, according as the given part opposite to it is greater or less than 90°.

In the third and fourth cases of the first form, and the first and second of the second form, the part required is greater or less than 90°, according as the negative term is greater or less than the affirmative.

ESTIMATION OF ERRORS OR SMALL QUANTITIES.

366. FOUR parts of a spherical triangle may be supposed to vary in magnitude, while the other two remain fixed. And it is frequently of use to know the relations of the increments or decrements of the variable parts, The methods of ascertaining these is what we are now to point out.

I. IN RIGHT-ANGLED TRIANGLES.

Let D, A, M, a, d represent as before, [323.] the five circular parts; A, D being on one side of the middle part M, and a, d on the other. Besides the right angle, let the part M remain invariable, and let the increments of the four variable parts, A, D, a, d be denoted by A', D', &c. (or their decrements by — A', — D', &c.); and their very small increments (say less than a minute) by A'', D'', &c.

Because cot A × cot a (= R . cos M) is of an invariable magnitude, tan A . tan a will also be invariable. Hence, $\tan(A+A') . \tan(a-a') = \tan A . \tan a$, and $\tan(A + A') : \tan A :: \tan a : \tan(a - a')$. Hence,

367. $\text{Sin } A' : \sin a' :: \sin(2A + A') : \sin(2a - a')$.

368. $A'' : a'' :: \sin 2A : \sin 2a :: \sin d : \sin D$. [321, 323.]

Again, sin D . sin d (= R . cos M) being also invariable, $\sin(D + D') : \sin D :: \sin d : \sin(d - d')$. Hence,

369. $\text{Tan } \frac{1}{2}D' : \tan \frac{1}{2}d' :: \tan(D+\frac{1}{2}D') : \tan(d-\frac{1}{2}d')$.

370. $D'' : d'' :: \tan D : \tan d :: \sin D : \cot A$.

Further, cos A . tan D (= R . cot M) is likewise invariable; therefore

cos (A+A') : cos A : : tan D : tan (D+D'). Hence

371. Tan $\frac{1}{2}$A' : sin D' : : cot (A+$\frac{1}{2}$A') : sin (2D+D').

372. A'' : D'' : : 2cot A : sin 2D : : tan d : sin D.

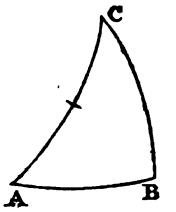
Lastly, because the ratio sin A : cos d (: : R : sin M) is invariable, sin (A+A') : cos (d—d') : : sin A : cos d.

373. Tan $\frac{1}{2}$A' : tan $\frac{1}{2}d'$: : tan (A+$\frac{1}{2}$A') : cot (d—$\frac{1}{2}d'$).

374. A'' : d'' : : tan A : cot d : : R : cos D.

These general principles applied to the several cases of a right-angled triangle, give the following theorems:

Case 1. The hypotenuse AC invariable.



375. Sin A' : sin C' : : sin (2A+A') : sin (2C — C').

376. A'' : C'' : : sin 2A : sin 2C : : cos AB : cos BC.

377. Tan$\frac{1}{2}$A' : sin AB' : : cot(A+$\frac{1}{2}$A') : sin(2AB–AB').

378. A'' : AB'' : : 2cot A : sin 2AB : : cot BC : cos AB.

379. Tan$\frac{1}{2}$AB' : tan$\frac{1}{2}$BC' : : tan(BC+$\frac{1}{2}$BC') : tan(AB–$\frac{1}{2}$AB').

380. AB'' : BC'' : : tan BC : tan AB : : cos AB : cot A.

381. Tan$\frac{1}{2}$BC' : tan$\frac{1}{2}$A' : : tan(BC+$\frac{1}{2}$BC') : tan(A+$\frac{1}{2}$A').

382. BC'' : A'' : : tan BC : tan A : : sin AB : R.

CASE II. The side BC invariable.

383. Sin AB' : sin C' : : sin (2AB + AB') : sin (2C + C').

384. AB'' : C'' : : sin 2AB : sin 2C : : sin AC : sin A.

385. Tan $\frac{1}{2}$AC' : tan $\frac{1}{2}$A' : : tan (AC + $\frac{1}{2}$AC') : tan (A − $\frac{1}{2}$A').

386. AC'' : A'' : : tan AC : tan A : : sin AC : cot C.

387. Tan $\frac{1}{2}$C' : sin AC' : : cot (C + $\frac{1}{2}$C') : sin (2AC + AC').

388. C'' : AC'' : : 2cot C : sin 2AC : : tan A : sin AC.

389. Sin A' : tan $\frac{1}{2}$AB' : : sin (2A − A') : tan (AB + $\frac{1}{2}$AB').

390. A'' : AB'' : : sin 2A : 2tan AB : : sin A : tan AC.

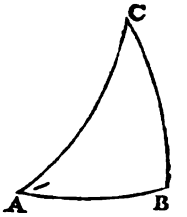
391. Tan $\frac{1}{2}$AB' : tan $\frac{1}{2}$AC' : : tan (AC + $\frac{1}{2}$AC') : tan (AB + $\frac{1}{2}$AB').

392. AB'' : AC'' : : tan AC : tan AB : : R : cos A.

393. Tan $\frac{1}{2}$A' : tan $\frac{1}{2}$C' : : cot (A — $\frac{1}{2}$A') : tan C + $\frac{1}{2}$C').

394. A'' : C'' : : cot A : tan C : : cos AC : R.

CASE III. The angle A invariable.



395. Sin AC' : sin AB' : : sin (2AC + AC') : sin (2AB + AB').

396. AC'' : AB'' : : sin 2AC : sin 2AB : : cos BC : sin C.

397. Tan $\frac{1}{2}$BC' : tan $\frac{1}{2}$C' : : cot (BC + $\frac{1}{2}$BC' : tan (C + $\frac{1}{2}$C').

398. BC'' : C'' : : cot BC : tan C : : cos BC : tan AB.

399. Tan $\frac{1}{2}$AC' : sin C' : : cot (AC + $\frac{1}{2}$AC') : sin (2C + C').

400. $AC'' : C'' :: 2\cot AC : \sin 2C :: \cot BC : \sin C.$

401. $\text{Tan}\frac{1}{2}AB' : \sin BC' :: \tan(AB+\frac{1}{2}AB') : \sin(2BC+BC').$

402. $AB'' : BC'' :: 2\tan AB : \sin 2BC :: \tan C : \cos BC.$

403. $\text{Tan}\frac{1}{2}AC' : \tan\frac{1}{2}BC' :: \tan(AC+\frac{1}{2}AC') : \tan(BC+\frac{1}{2}BC').$

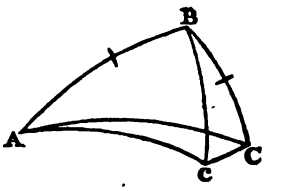
404. $AC'' : BC'' :: \tan AC : \tan BC :: R : \cos C.$

405. $\text{Tan}\frac{1}{2}AB' : \tan\frac{1}{2}C' :: \tan(AB+\frac{1}{2}AB') : \tan(C+\frac{1}{2}C').$

406. $AB'' : C'' :: \tan AB : \tan C :: R : \sin BC.$

II. IN OBLIQUE-ANGLED TRIANGLES.

CASE I. When the two sides AB, BC are invariable.



Since $\sin A : \sin C$ ($::\sin BC : \sin AB$) $::$ $\sin(A + A') : \sin(C + C')$.

407. $\text{Tan}\frac{1}{2}A' : \tan\frac{1}{2}C' :: \tan(A+\frac{1}{2}A') : \tan(C+\frac{1}{2}C').$

408. $A'' : C'' :: \tan A : \tan C.$

Let the side BC move into the position Bc, that is, let CBc ($= -\frac{1}{2}B'$) be the variation of the angle B, then because the angle $BCc = BcC$,

$$AcC - ACc = ACB + AcB = 2C + C';$$

hence,

409. $\sin\frac{1}{2}AC' : \tan\frac{1}{2}A' :: \sin(AC - \frac{1}{2}AC') : \cot(C + \frac{1}{2}C')$. [284.]

410. $AC'' : A'' :: \sin AC : \cot C :: \sin AB . \sin B : R . \cos C$.

In the isosceles triangle BCc, $R . \sin\frac{1}{2}Cc = \sin BC . \sin\frac{1}{2}B'$, [301.] hence, from the triangle ACc, we have, [294.]

411. $\sin\frac{1}{2}A' : \sin\frac{1}{2}B' :: \sin BC . \cos(C + \frac{1}{2}C') : R . \sin(AC - \frac{1}{2}AC')$.

412. $A'' : B'' :: \sin BC . \cos C : R . \sin AC :: \sin A . \cos C : R . \sin B :: R^2 . \cos AB - \sin AB . \cos A . \cot AC : R^2$.

In like manner,

$R . \cos\frac{1}{2}A' : \sin BC . \sin\frac{1}{2}B' :: \sin(C + \frac{1}{2}C') : \sin\frac{1}{2}AC'$,

hence,

413. $\sin\frac{1}{2}B' : \sin\frac{1}{2}AC' :: R . \cos\frac{1}{2}A' : \sin BC . \sin(C + \frac{1}{2}C')$.

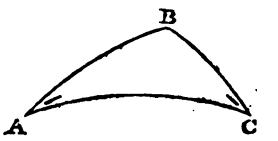
414. $B'' : AC'' :: R^2 : \sin BC . \sin C :: R^2 : \sin A . \sin AB :: R^2 . \sin AC : \sin AB . \sin BC . \sin B$.

Lastly, because $R^2 . \cos AC - \cos B . \sin AB . \sin BC (= R . \cos AB . \cos BC) = R^2 . \cos(AC - AC') - \cos(B - B') . \sin AB . \sin BC$, therefore, [24.]

415. $\sin\frac{1}{2}AC' : \sin\frac{1}{2}B' :: \sin AB . \sin BC . \sin(B - \frac{1}{2}B') : R^2 . \sin(AC - \frac{1}{2}AC')$.

416. $\sin\frac{1}{2}A' : \sin\frac{1}{2}AC' :: R . \cos(C + \frac{1}{2}C') : \sin AB . \sin(B - \frac{1}{2}B')$. [411, 415.]

CASE II. When the angles A, C are invariable.



417. Tan $\frac{1}{2}AB'$: tan $\frac{1}{2}BC'$: : tan $(AB + \frac{1}{2}AB')$: tan $(BC + \frac{1}{2}BC')$.

418. AB'' : BC'' : : tan AB : tan BC.

419. Sin $\frac{1}{2}B'$: tan $\frac{1}{2}AB'$: : sin $(B + \frac{1}{2}B')$: cot $(BC + \frac{1}{2}BC')$.

420. B'' : AB'' : : sin B : cot BC.

421. Sin $\frac{1}{2}AB'$: sin $\frac{1}{2}AC'$: : sin C . cos $(BC + \frac{1}{2}BC')$: R . sin $(B + \frac{1}{2}B')$.

422. AB'' : AC'' : : sin C . cos BC : R . sin B : : sin AB . cos BC : R . sin AC : : R . cos A + cos B . cos C : $\sin^2 B$.

423. Sin $\frac{1}{2}AC'$: sin $\frac{1}{2}B'$: : R . cos $\frac{1}{2}AB'$: sin C . sin $(BC + \frac{1}{2}BC')$.

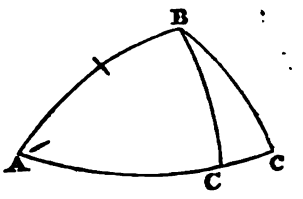
424. Sin $\frac{1}{2}AC'$: sin $\frac{1}{2}B'$: : R^2 . sin $(B + \frac{1}{2}B')$: sin A . sin C . sin $(AC + \frac{1}{2}AC')$.

425. AC'' : B'' : : R^2 : sin C . sin BC : : R^2 : sin A . sin AB.

426. Sin $\frac{1}{2}AB'$: sin $\frac{1}{2}B'$: : R . cos $(BC + \frac{1}{2}BC')$: sin A . sin $(AC + \frac{1}{2}AC')$.

These theorems may be obtained by applying those in the preceding case to the reciprocal triangle.

CASE III. When the side AB, and the adjacent angle A are invariable.



$\sin BC.\sin C(=\sin AB.\sin A) = \sin(BC+BC').\sin(C-C')$.

Hence,

427. $\tan\frac{1}{2}BC' : \tan\frac{1}{2}C' :: \tan(BC+\frac{1}{2}BC') : \tan(C-\frac{1}{2}C')$.

428. $BC'' : C'' :: \tan BC : \tan C$.

From the triangle CBc, we have,

429. $\tan\frac{1}{2}C' : \tan\frac{1}{2}B' :: \cos(BC+\frac{1}{2}BC') : \cos\frac{1}{2}BC'$. [285.]

430. $\sin\frac{1}{2}C' : \sin\frac{1}{2}B' :: \cos(BC+\frac{1}{2}BC') : \cos\frac{1}{2}AC'$. [295.]

431. $C'' : B'' :: \cos BC : R$.

432. $\sin B' : \sin AC' :: \sin C : \sin(BC+BC')$. [260.]

433. $\sin\frac{1}{2}B' : \sin\frac{1}{2}AC' :: \sin(C-\frac{1}{2}C') : \sin(BC+\frac{1}{2}BC')$. [294.]

434. $B'' : AC'' :: \sin C : \sin BC :: \sin^2 C : \sin A . \sin AB$.

435. $\sin\frac{1}{2}BC' : \tan\frac{1}{2}B' :: \sin(BC+\frac{1}{2}BC') : \tan(C-\frac{1}{2}C')$. [384.]

436. $\sin\frac{1}{2}C' : \tan\frac{1}{2}AC' :: \sin(C-\frac{1}{2}C') : \tan(BC+\frac{1}{2}BC')$. [258.]

437. $C'' : AC'' :: \sin C : \tan BC$.

438. Tan $\frac{1}{2}$BC$'$: tan $\frac{1}{2}$AC$'$: : cos (C — $\frac{1}{2}$C$'$) : cos $\frac{1}{2}$C$'$. [257.]

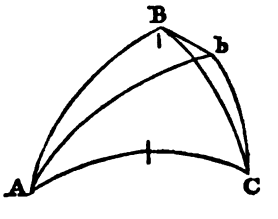
439. BC$''$: AC$''$: : cos C : R.

Lastly, by applying the general theorems, [346,—349.] to the triangle AB*c*, we have,

440. Cot BC . sin (B + B$'$) = cot AB . sin B$'$ + cot (BC + BC$'$) . sin B.

&c. &c.

Case iv. When the side AC, and the opposite angle B are invariable.



Sin A : sin BC (: : sin B : sin AC) : : sin (A—A$'$) : sin (BC—BC$'$).

441. Tan $\frac{1}{2}$A$'$: tan $\frac{1}{2}$BC$'$: : tan (A—$\frac{1}{2}$A$'$) : tan (BC—$\frac{1}{2}$BC$'$).

442. A$''$: BC$''$: : tan A : tan BC.

From the triangles AB*b*, CB*b*, (the sum of the angles at the common base in each being the same,) we have,

443. Sin $\frac{1}{2}$C$'$: sin $\frac{1}{2}$A$'$: : cos (AB + $\frac{1}{2}$AB$'$) : cos (BC—$\frac{1}{2}$BC$'$). [295.]

444. Cos $\frac{1}{2}$C$'$: cos $\frac{1}{2}$A$'$: : cos $\frac{1}{2}$AB$'$: cos $\frac{1}{2}$BC$'$. [293.]

445. C$''$: A$''$: : cos AB : cos BC.

Again, $\sin(AB+AB') : \sin(C+C') :: \sin BC : \sin A$ :: sin AB : sin C :: sin (BC — BC') : sin (A — A').
Therefore, $\cos(AB+\frac{1}{2}AB') . \sin\frac{1}{2}AB' : \cos(C+\frac{1}{2}C').\sin\frac{1}{2}C'$ :: cos (BC—$\frac{1}{2}$BC') . sin $\frac{1}{2}$BC' : cos (A—$\frac{1}{2}$A').sin$\frac{1}{2}$A'.
Hence,

446. $\text{Sin}\frac{1}{2}AB' : \sin\frac{1}{2}BC' :: \cos(C+\frac{1}{2}C') : \cos(A—\frac{1}{2}A')$. [443.]

447. AB'' : BC'' : : cos C : cos A.

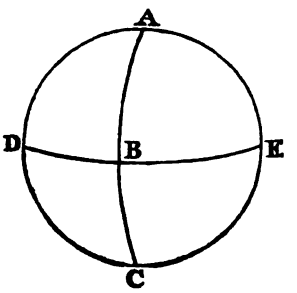
448. Sin $\frac{1}{2}$BC' : sin $\frac{1}{2}$A' : : sin BC . cos (A — $\frac{1}{2}$A') : sin A . cos (BC + $\frac{1}{2}$BC').

449. Sin $\frac{1}{2}$AB' : sin $\frac{1}{2}$A' : : sin BC . cos (C + $\frac{1}{2}$C') : sin A . cos (BC + $\frac{1}{2}$BC').

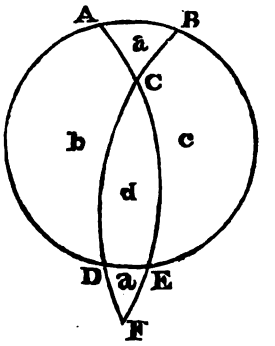
450. AB'' : A'' : : sin BC . cos C : sin A . cos BC : : tan BC . cos C : R . sin A.

If only one part of a spherical triangle continue fixed, while each of the remaining parts varies by a very small quantity, (as a few seconds); then, any two of these variations being given, the other three may be found.

TO MEASURE THE SURFACE OF A SPHERICAL TRIANGLE.



LET ADCE be the base of a hemisphere, ABC any great circle meeting it in A, C, and let A, C be the poles of the great circle DBE. Then, if the point B be conceived to move uniformly along the arc DBE, the spaces described by B and ABC being both proportional to the times, are proportional to one another: that is, the lunary space ADCBA : the lunary space ABCEA : : arc DB : BE : : angle DAB : BAE. Hence, if S denote the surface of the sphere, L any lunary surface, and A the angle contained by the semicircles bounding it,



451. $180^{\circ} : A :: \frac{1}{2}S : L.$

Again, let ABC be any spherical triangle, and let the circle, of which the side AB is an arc, be completed: Also, let the sides AC, BC be produced, to intersect

the former in E, D, and to intersect each other again in F: The triangles ABC, EDF, on opposite sides of the sphere, are equivalent; for it is manifest, that the sides and angles of the one are equal to those of the other, each to each. Now, if a, b, c, d denote each the triangular space in which it is placed,

$$180^\circ : A :: \tfrac{1}{2}S : a + c. \quad [451.]$$
$$180^\circ : B :: \tfrac{1}{2}S : a + b.$$
$$180^\circ : C :: \tfrac{1}{2}S : a + d.$$

Therefore $180^\circ : A + B + C :: \frac{1}{2}S : 3a + b + c + d.$

452. And $180^\circ : A + B + C - 180^\circ :: \frac{1}{2}S : 2a :: \frac{1}{2}S : a$; that is, as two right angles are to the excess of the angles of a spherical triangle above two right angles, so is the area of a great circle to the area of the triangle.

Let E denote the excess of the three angles above 180°, and R° the radius of the sphere in degrees; then, since $180^\circ : E :: 180 . R^\circ :$ Area,

453. The area of the triangle in square degrees $= E . R^\circ$.

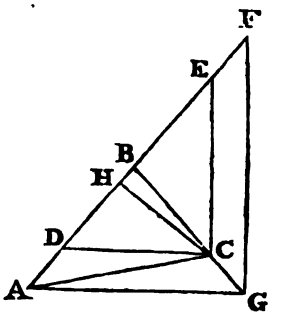
ANALOGY BETWEEN PLANE AND SPHERICAL TRIANGLES.

ALTHOUGH the sum of the angles of a spherical triangle is not, as the sum of the angles of a plane tri-

angle, of an invariable magnitude, yet this circumstance does not render any case of a spherical triangle more indeterminate than the corresponding case of a plane triangle. On the contrary, one case (where the three angles are given,) is perfectly limited in the former species of triangles, while it remains indeterminate in the latter. There is, however, a striking analogy between the theorems respecting each, which we shall now illustrate, by deducing, in a similar manner, from one general proposition the theorems of plane trigonometry.

Let ABC be a plane triangle, of which the side AB is greater than BC. Produce AB above the vertex to F, and BC below the base to G, so that BF=BG=BA.

Also, make BE=BC=BD, join AG, GF, DC, CE, and let fall the perpendicular CH upon AB. The triangles AGF, DCE are evidently right-angled and similar; hence AF.DE: AG.DC: : DE^2 : DC^2 : : DE : DH, that is, 4AB.BC : AC^2 — AD^2 : : 2R : vers B.



Thus, putting AB + BC = S, AB — BC = D, and $\frac{1}{2}$AB + $\frac{1}{2}$BC + $\frac{1}{2}$CA = P, we have,

454. $2AB \cdot BC : AC^2 - D^2 :: R : \text{vers } B.$

455. $2AB \cdot BC : S^2 - AC^2 :: R : \text{suvers } B.$

456. $2AB \cdot BC : AB^2 + BC^2 - AC^2 :: R : \cos B.$

457. $AB \cdot BC : \frac{1}{2}(AC+D)\frac{1}{2}(AC-D) :: R^2 : \sin^2\frac{1}{2}B.$

458. $AB \cdot BC : (P - BC)(P - AB) :: R^2 : \sin^2\frac{1}{2}B.$

459. $AB \cdot BC : \frac{1}{2}(S+AC)\frac{1}{2}(S-AC) :: R^2 : \cos^2\frac{1}{2}B.$

460. $AB \cdot BC : P \cdot (P - AC) :: R^2 : \cos^2\frac{1}{2}B.$

461. $P \cdot (P-AC) : (P-BC)(P-AB) :: R^2 : \tan^2\frac{1}{2}B.$

462. $\text{Tan}\frac{1}{2}A : \tan\frac{1}{2}C :: P-AB : P-BC :: AC-D : AC+D.$

463. $\text{Cot}\frac{1}{2}A : \tan\frac{1}{2}C :: P : P-AC :: S+AC : S-AC.$

464. $\text{Sin}\frac{1}{2}(C+A) : \sin\frac{1}{2}(C - A) :: AC : D.$
or, $\cos\frac{1}{2}B : \cos(A + \frac{1}{2}B) :: AC : D.$

465. $\text{Cos}\frac{1}{2}(C + A) : \cos\frac{1}{2}(C - A) :: AC : S.$
or, $\sin\frac{1}{2}B : \sin(A + \frac{1}{2}B) :: AC : S.$

466. $\text{Tan}\frac{1}{2}(C + A) : \tan\frac{1}{2}(C - A) :: S : D.$
or, $\tan\frac{1}{2}B : \tan(A + \frac{1}{2}B) :: D : S.$

467. $\text{Sin } C : \sin A :: AB : BC.$

468. $2AB.AC : \sqrt{(AC^2-D^2)(S^2-AC^2)} :: R : \sin A.$

---

469. $\text{Sin } B : \sin(C - A) :: AC^2 : S \cdot D.$

470. $\text{Tan } A : \sin B \left(:: \frac{1}{AB} : \frac{1}{BC} - \frac{1}{AB} \cdot \cos B\right) ::$
$BC : AB - BC \cdot \cos B.$

471. $\frac{\text{Sin B}}{\text{BP}} = \frac{\sin \text{CBP}}{\text{AB}} + \frac{\sin \text{ABP}}{\text{BC}}$. (P any point in AC).

472. Cot P . AC = cot A . PC — cot C . AP.

473. 2CotP.sinA.sinC=sin(A⌢C), (when AP=PC.)

474. Cot P . (AB+BC) = cot A . BC — cot C . AB.

475. $2\text{Cos } \frac{1}{2}\text{B} \cdot \text{BP} = \frac{\text{AB} \cdot \text{BC}}{(\text{AB} + \text{BC})}$, when BP bisects the angle B.

476. $2\text{Sin } \frac{1}{2}\text{B} \cdot \text{BP} = \frac{\text{AB} \cdot \text{BC}}{(\text{AB} - \text{BC})}$, when it bisects the angle adjacent to B.

477. $\begin{cases} \text{AB} : \text{BC} :: \text{R} : \tan x. \\ \text{R} : \tan (45 + x) :: \tan \frac{1}{2}\text{B} : \tan (\text{A} + \frac{1}{2}\text{B}). \end{cases}$

478. $\begin{cases} \text{AB} : \text{BC} :: \sin \text{B} : \tan x. \\ \text{Sin } (\text{B} \frown x) : \sin x :: \sin \text{B} : \tan \text{A}. \end{cases}$

479. $\begin{cases} \text{S} : \text{AC} :: \text{R} : \tan x. \\ \text{AC} : \text{D} :: \text{R} : \tan y. \\ \text{Tan}^2 \frac{1}{2}\text{A} = \cot (45 + x) \cdot \cot (45 + y). \end{cases}$

480. BC : AB :: sin A : sin (A + B) :: cosec B : cot A + cot B.

&c. &c.

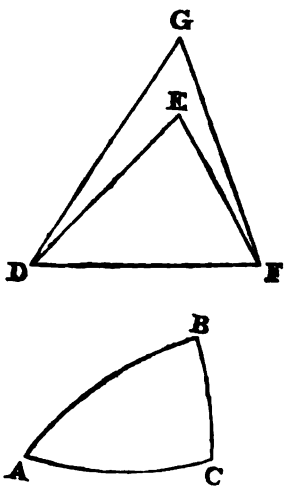
In general, the theorems respecting plane triangles may be deduced from those of spherical triangles, by

this consideration, That when an arc is such a part of the whole circumference as to differ insensibly from the straight line between the same points, its sine and tangent are each equal to that straight line, and its cosine equal to the radius.

REDUCTION OF SPHERICAL TO PLANE TRIANGLES.

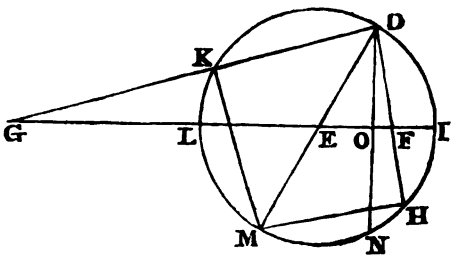
LET the two sides AB, BC, and the included angle B, of the spherical triangle ABC, be given, to find, by construction, on a plane surface, the side AC, and the angles A, C.

481. 1. To find the base AC.



Make the angle DEF=B, DE = tan AB, and EF = tan BC; also, make DG = sec AB, and FG = sec BC; then the arc which measures the angle DGF, to the same radius, is equal to AC. This appears by projecting the spherical triangle upon a plane touching the sphere in the vertex of the triangle, the centre of the sphere being the projecting point. (STEREOGRAPHIC PROJECTION.)

482. 2. To find the angles A, C.



Make the angle DEF = B, DE = tan $\frac{1}{2}$AB, EF = tan $\frac{1}{2}$BC, and in FE produced take EG = cot $\frac{1}{2}$BC. About the centre E, with the radius ED, describe the circumference of a circle to intersect the straight lines DF, DG, EF in H, K, I; then IH measures the difference, and IDK the sum of the angles A, C.

For, let DON be perpendicular to the diameter IEL; then, because DE : EF : : sin EFD : sin EDF : : cos ODF : cos HMD, tan $\frac{1}{2}$AB : tan $\frac{1}{2}$BC : : cos ($\frac{1}{2}$B — $\frac{1}{2}$IH) : cos ($\frac{1}{2}$B + $\frac{1}{2}$IH).

Consequently IH = C — A. [279.]

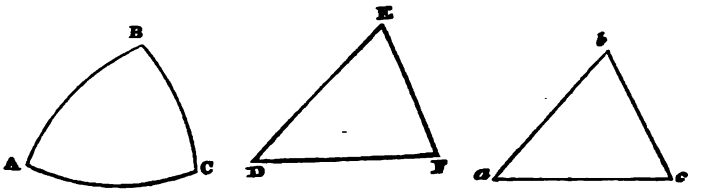
In like manner, because GE : ED : : sin GDE : sinG : : cosKMD : cosODG: : cos$\frac{1}{2}$KD : cos$\frac{1}{2}$KDIN; therefore (cot $\frac{1}{2}$BC : tan $\frac{1}{2}$AB : :) cot $\frac{1}{2}$AB : tan $\frac{1}{2}$BC : : cos ($\frac{1}{2}$IDK — $\frac{1}{2}$B) : cos ($\frac{1}{2}$IDK + $\frac{1}{2}$B).

Consequently IDK = C + A. [280.]

483. Cor. 1. Hence, the angle C = EFD + G, and A = EDF + G.

484. Cor. 2. The angle G (or IDF) is half the excess of the sum of the angles A, B, C above two right angles.

485. Cor. 3. Hence, putting $A+B+C-180^\circ=s$, and supposing the arcs AB, BC so small, that $\frac{1}{2}AB$ and $\frac{1}{2}BC$ may be reckoned equal to their tangents; then, if the plane angle DEF be made equal to the spherical angle B, and the straight lines DE, EF, that contain the former, be made equal to the arcs AB, BC respectively, which contain the latter, the angles at the base DF, of the plane triangle DEF, will be such, that $EDF = A - \frac{1}{2}s$, and $EFD = C - \frac{1}{2}s$.



486. Cor. 4. Hence the following theorem, (discovered by Legendre):

If the sides of the spherical triangle ABC, (each not exceeding a degree,) be extended into the straight lines which form the plane triangle *abc*; then will each angle of the latter triangle be less than the corresponding angle of the former by $\frac{1}{3}s$, or one-third of the excess of the three spherical angles above two right angles.

For, let the straight lines DE, EF be equal to the arcs AB, BC, (or to the straight lines, *ab*, *bc*,) and the angle E equal to B, and let DF be joined; then, because,

Sin a : sin c : : cb : ba : : FE : ED : : sin (A — $\frac{1}{2}s$) : sin (C — $\frac{1}{2}s$), [488.]

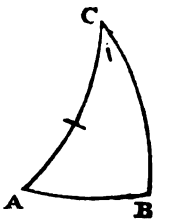
and, in like manner,

Sin c : sin b : : sin (C — $\frac{1}{2}s$) : sin (B — $\frac{1}{2}s$),

the sines of the angles a, c, b are proportional to the sines of (A — $\frac{1}{2}s$), (C — $\frac{1}{2}s$), (B — $\frac{1}{2}s$); but, since the arcs CB, BA, AC are so small as to be equal to their sines, the sines of a, c, b are also proportional to the sines of A, C, B; therefore the sines of a, c, b are not only proportional, but equal to the sines of the intermediate angles (A—$\frac{1}{3}s$), (C—$\frac{1}{3}s$), (B—$\frac{1}{3}s$), whereof the sum is two right angles.

PRACTICAL SOLUTION OF THE CASES OF A RIGHT ANGLED SPHERICAL TRIANGLE.

CASE I. Given the hypotenuse AC, 52° 30′, and the angle C, 34° 16′.



1. To find the side AB opposite to the given angle.

487. { Sin AB = sin C. sin AC. } Sin C..... 9.7505434
{ AB, C are like. } Sin AC .. 9.8994667

AB = 26° 31′ 54″ . 37.....Sin AB 9.6500101

Or thus, by natural sines,

488. $\left\{\begin{array}{c} 2\sin AB = \cos(AC-C) - \cos(AC+C). \\ AB,\ C \text{ like.} \end{array}\right\}$ $\left\{\begin{array}{l} \text{Diff. n. cos } .9497902 \\ \text{Sum n. cos } .0564024 \\ \hline 2).8933878 \end{array}\right.$

$AB = 26^\circ\ 31'\ 54''.35$ ..... n. sin AB .4466939

2. To find the other side BC.

489. $\left\{\begin{array}{c} \text{Tan BC} = \cos C \cdot \tan AC. \\ \text{If } AC \lessgtr 90^\circ, \text{ BC, C like, if } >, \text{ unlike.} \end{array}\right\}$ $\left\{\begin{array}{ll} \text{Cos C} & 9.9172040 \\ \text{Tan AC} & 10.1150195 \end{array}\right.$

$BC = 47^\circ\ 7'\ 25''.14$ ..... Tan BC 10.0322235

3. To find the other angle A.

490. $\left\{\begin{array}{c} \text{Tan A} = \cot C \cdot \sec AC. \\ \text{If } AC \lessgtr 90^\circ, \text{ A, C like, if } >, \text{ unlike.} \end{array}\right\}$ $\left\{\begin{array}{ll} \text{Cot C} & 10.1666606 \\ \text{Sec AC} & 0.2155529 \end{array}\right.$

$A = 67^\circ\ 28'\ 25''.56$ ..... Tan A 10.3822135

4. To find both BC and A.

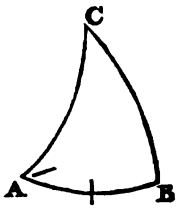
491. $\left\{\begin{array}{l} \text{Tan}\frac{1}{2}(A+BC) = \cot(45+\frac{1}{2}C).\tan(45+\frac{1}{2}AC) \\ \text{Tan}\frac{1}{2}(A-BC) = \cot(45+\frac{1}{2}C).\cot(45+\frac{1}{2}AC) \end{array}\right.$

	$\frac{1}{2}C.$	$\frac{1}{2}AC.$
	17° 8′	26° 15′
	45	45
	62 8	71 15

62° 8′ cot 9.7232324 ..... 9.7232324 | $\frac{1}{2}$sum 57° 17′ 55″.33
71 15 tan 10.4692187 ..... cot 9.5307813 | $\frac{1}{2}$diff. 10 10 30 .21

$\frac{1}{2}$Sum tan 10.1924511 . $\frac{1}{2}$diff. tan 9.2540137 $A = 67\ 28\ 25\ .54$
$BC = 47\ 7\ 25\ .12$

CASE II. Given the side AB, 62° 25′, and its adjacent angle A, 58° 8′.



1. To find the other angle C.

492. $\left\{\begin{array}{l}\text{Cos C} = \sin \text{A} \cdot \cos \text{AB}. \\ \text{C, AB like.}\end{array}\right\}$ Sin A 9.9031084
Cos AB 9.6656168

C = 68° 15′ 24″ . 70 Cos C 9.5687252

Or thus, by natural sines,

493. 2Cos C = sin(AB + A) — sin(AB — A). 9022092
1613167

2)7408925

C = 68° 15′ 24″ . 7 ... nat. cos C 3704462½

2. To find the hypotenuse AC.

494. $\left\{\begin{array}{l}\text{Tan AC} = \sec \text{A} \cdot \tan \text{AB}. \\ \text{If AB, A like, AC} \lessgtr 90°, \text{ if not, } \gtrless.\end{array}\right|$ Sec A 0.2218814
Tan AB 10.2819827

AC = 72° 35′ 51″ . 42 ..... Tan AC 10.5038641

3. To find the remaining side BC.

495. $\left\{\begin{array}{l}\text{Tan BC} = \tan \text{A} \cdot \sin \text{AB}. \\ \text{BC, A like.}\end{array}\right\}$ Tan A 10.1249898
Sin AB 9.9475995

BC = 49° 45′ 58″ . 15 Tan BC 10.0725893

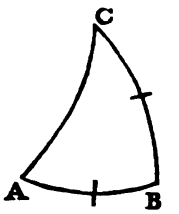
4. To find both AC and CB.

496. $\left\{\begin{array}{l}\text{Tan}\frac{1}{2}(AC+CB)=\tan\frac{1}{2}AB.\tan(45+\frac{1}{2}A).\\ \text{Tan}\frac{1}{2}(AC-CB)=\tan\frac{1}{2}AB.\cot(45+\frac{1}{2}A).\end{array}\right|$ If $A>90°$, BC $>$ AC, if $\angle$, $\angle$.

Tan $\frac{1}{2}$AB	9.7823438	.....	9.7823438
Tan (45 + $\frac{1}{2}$A)	10.4771621	..... cot	9.5228379
61° 10′ 54″.79..T.	10.2595059	11° 24′ 56″.63..T.	9.3051817

$\frac{1}{2}$sum	61°	10′	54″	. 79
$\frac{1}{2}$diff.	11	24	56	. 63
AC =	72	35	51	. 42
CB =	49	45	58	. 16

CASE III. Given the side AB, 48° 22′, and the side BC, 31° 15′.



1. To find the hypotenuse AC.

497. $\left\{\begin{array}{l}\text{Cos AC} = \cos AB \cdot \cos BC.\\ \text{If AB, BC like, AC} < 90°, \text{ if not, } >.\end{array}\right|$ Cos AB 9.8224042; Cos BC 9.9319213

AC = 55° 23′ 28″ . 33 ..... Cos AC 9.7543255

Or thus, by natural sines,

498. $\left\{\begin{array}{l}2\text{CosAC}=\cos(AB+BC)+\cos(AB-BC)\\ \text{Limits as above.}\end{array}\right|$ 1802330; 9557074; 2)11359404

AC=55° 23′ 28″ . 33 ..... nat. cos AC 5679702

2. To find one of the angles, as C.

499. $\left\{\begin{array}{c}\text{Tan C} = \text{cosec BC . tan AB.} \\ \text{C, AB like.}\end{array}\right\}$ $\left\{\begin{array}{ll}\text{Cosec BC} & 0.2850224 \\ \text{Tan AB} & 10.0511557\end{array}\right.$

$C = 65° 14' 39''.22$ ... Tan C 10.3361781

3. To find both angles A, C.

500. $\left\{\begin{array}{l}\text{Tan}\frac{1}{2}(A+C) = \text{sec } \frac{1}{2}(AB+BC).\cos\frac{1}{2}(AB-BC) \\ \text{Tan}\frac{1}{2}(A-C) = \text{cosec}\frac{1}{2}(AB+BC).\sin\frac{1}{2}(AB-BC)\end{array}\right.$ $\left\{\begin{array}{l}\text{A, BC like.} \\ \text{C, AB like.}\end{array}\right.$

Sec	0.1145311	Cosec	0.1936698	½Sum	52°	9′	30″	.53
Cos	9.9951369	Sin	9.1726504	½Diff.	13	5	8	.67
Tan	10.1096680	Tan	9.3663202	C =	65	14	39	.20
				A =	39	4	21	.86

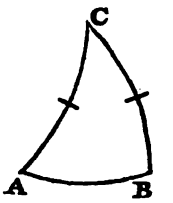
When AB, BC are both small arcs, and it is required to find AC with the utmost accuracy, the following method may be adopted:

Ex. Given AB 4° 1′ 13″, and BC 3° 55′ 17″ | Cagnoli, p. 303.

501. $\left\{\begin{array}{l}\text{Sin}\frac{1}{2}S : \sin\frac{1}{2}D :: R : \tan\frac{1}{2}d. \text{ [301.]} \\ \text{Sin}\frac{1}{2}d : \sin\frac{1}{2}D :: \sin 45° : \sin\frac{1}{2}AC. \text{ [312.]}\end{array}\right.$ $\left\{\begin{array}{l}\frac{1}{2}S = 3° 58′ 15″ \\ \frac{1}{2}D = 2\ 58\end{array}\right.$

Cosec ½S	1.1595887			
Sin ½D	6.9359948	.....	6.9359948	
Tan ½d	8.0955835	Sin 45°	9.8494850	
Diff. in tables	337	Cosec ½d	1.9044502	
Sin ½d	8.0955498	Sin ½AC	8.6899300	2° 48′ 24″.83
				2
			AC =	5 36 49 .66

CASE IV. Given the hypotenuse AC, 75° 13′, and the side CB, 43° 32′.



1. To find the other side AB.

502. { Cos AB = cos AC . sec BC. } Cos AC 9.4068203
{ If AC ∠ 90°, AB, BC like, if >, unlike. } Sec BC 0.1420922

AB = 69° 16′ 20″ . 50 ..... Cos AB 9.5489125

Or thus,

503. { Tan² ½AB = tan ½(AC + CB).tan ½(AC − CB) | 10.2505740
{ ½AB = 34° 38′ 10″ . 26 | 9.4481133
2 | 2)19.6786873

AB = 69 16 20 . 52 ..... Tan ½AB 9.8393436½

2. To find the included angle C.

504. { Cos C = cot AC . tan BC. } Cot AC 9.4214398
{ If AC ∠ 90°, C, BC like, if >, unlike. } Tan BC 9.9828145

C = 75° 18′ 20″ . 60 ..... Cos C 9.4042543

Or thus,

505. { Tan² ½C = cosec ½(AC + CB).sin ½(AC − CB) | 10.0585315
{ ½C = 37° 39′ 10″ . 32 | 9.7162243
2 | 2)19.7747558

C = 75 18 20 . 64 ..... Tan ½C 9.8873779

3. To find the other angle A.

506. $\left\{\begin{array}{l}\text{Sin A} = \text{cosec AC} \cdot \sin \text{BC} \\ \text{A, BC like.}\end{array}\right.$ $\left\{\begin{array}{ll}\text{Cosec AC} & 0.0146195 \\ \text{Sin BC} & 9.8407223 \\ \text{Sin A} & 9.8553418\end{array}\right.$

$\text{A} = 45^\circ\ 46'\ 59''.86 \ldots$.

Or thus,

507. $\left\{\begin{array}{l}\text{Tan}^2(45+\frac{1}{2}\text{A}) = \tan\frac{1}{2}(\text{AC}+\text{CB}) \\ \qquad\qquad \times \cot\frac{1}{2}(\text{AC}-\text{CB})\end{array}\right.$ $\left|\begin{array}{l}10.2305740 \\ 10.5518867 \\ 2)\overline{20.7824617}\end{array}\right.$

$(45+\frac{1}{2}\text{A}) = 67^\circ\ 53'\ 29''.91$ $\quad \text{Tan}(45+\frac{1}{2}\text{A})\ \ 10.3912308\frac{1}{2}$

$\text{A} = 45\ \ 46\ \ 59\ .\ \overset{2}{82}$

4. To find A, C and AB.

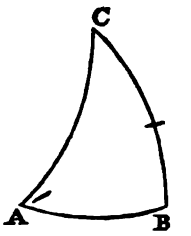
Put $\text{N} = \sqrt{\sin(\text{AC}+\text{CB}) \cdot \sin(\text{AC}-\text{CB})}$. Then,

508. $\left\{\begin{array}{ll}\text{TanAB} = \text{N} \cdot \sec \text{AC}. \\ \text{TanC} = \text{N} \cdot \sec\text{AC} \cdot \text{cosec BC}. \\ \text{CotA} = \text{N} \cdot \text{cosec BC}.\end{array}\right.$ $\left|\begin{array}{ll}\text{N} \ldots\ldots & 9.8288464 \\ \text{SecAC} & 0.5931797 \\ \text{TanAB} & 10.4220261\end{array}\right.$

9.9414685	AB = 69° 16′ 20″ . 50.	CosecBC	0.1592777
9.7162243	C = 75 18 20 . 60.	TanC	10.5813038
2)19.6576928	A = 45 46 59 . 83.	CotA	9.9881241

N = 9.8288464

CASE v. Given the side BC, 35° 18′, and its opposite angle A, 47° 28, AB being $\angle$ 90°.



1. To find the hypotenuse AC.

509. $\left\{\begin{array}{l} \text{Sin AC} = \text{cosec A} \cdot \text{sin BC}, \\ \text{If AB, BC like, AC} \lessgtr 90°, \text{ if not, } \gtrless. \end{array}\right.$ $\left\{\begin{array}{ll} \text{CosecA} & 0.1326008 \\ \text{SinBC} & 9.7618208 \end{array}\right.$

AC = 51° 38′ 45″ . 15 ..... SinAC 9.8944216

Or thus,

510. $\left\{\text{Tan}^2(45+\frac{1}{2}\text{AC}) = \tan\frac{1}{2}(\text{A}+\text{BC}) \times \cot\frac{1}{2}(\text{A} \sim \text{BC})\right\}$ $\left\{\begin{array}{lll} 41° & 23′ & \tan \ 9.9450261 \\ 6 & 5 & \cot \ 10.9723448 \\ & & 2)\overline{20.9173709} \end{array}\right.$

$(45+\frac{1}{2}\text{AC}) = 70° 49′ 22″ . 58$ $\tan(45+\frac{1}{2}\text{AC})$ $10.4586854\frac{1}{2}$

AC = 51 38 45 . 16

2. To find the other side AB.

511. $\left\{\begin{array}{l} \text{Sin AB} = \text{cot A} \cdot \text{tan BC}. \\ \text{AB supposed} \lessgtr 90°. \end{array}\right.$ $\left\{\begin{array}{ll} \text{Cot A} & 9.9625597 \\ \text{Tan BC} & 9.8500575 \end{array}\right.$

AB = 40° 30′ 29″ . 51 Sin AB 9.8126172

Or thus,

512. $\left\{\text{Tan}^2(45+\frac{1}{2}\text{AB}) = \text{cosec}(\text{A} \sim \text{BC}) \times \sin\ (\text{A}+\text{BC})\right\}$ $\left\{\begin{array}{lll} 12° & 10′ & \text{cosec } 10.6762198 \\ 82 & 46 & \sin \ 9.9965299 \\ & & 2)\overline{20.6727497} \end{array}\right.$

$(45+\frac{1}{2}\text{AB}) = 65° 15′ 14″ . 76$ $\tan(45+\frac{1}{2}\text{AB})$ $10.3363748\frac{1}{2}$

AB = 40 30 29 . 52

3. To find the other angle C.

513. $\left\{\begin{array}{c}\text{Sin C} = \sec \text{BC} \cdot \cos \text{A.} \\ \text{C, AB like.}\end{array}\right\}$ $\left\{\begin{array}{ll}\text{Sec BC} & 0.0882366 \\ \text{Cos A} & 9.8299589\end{array}\right.$

$\text{C} = 55^\circ\ 55'\ 33''.71$ Sin C 9.9181955

Or thus:

514. $\left\{\begin{array}{c}\text{Tan}^2(45+\frac{1}{2}\text{C}) = \cot\frac{1}{2}(\text{A}+\text{BC}) \\ \times \cot\frac{1}{2}(\text{A}-\text{BC}).\end{array}\right\}$ $\left\{\begin{array}{ll}41^\circ\ 23' \cot & 10.0549739 \\ 6^\circ\ 5' \cot & 10.9723448 \\ & 2)\overline{21.0273187} \\ \text{Tan}(45+\frac{1}{2}\text{C}) & \overline{10.5136593}\frac{1}{2}\end{array}\right.$

$(45+\frac{1}{2}\text{C}) = 72^\circ\ 57'\ 46''.86$

$\text{C} = 55^\circ\ 55'\ 33''.72$

4. To find AC, AB and C.

Put $n = \sqrt{\sin(\text{A}+\text{BC}) \cdot \sin(\text{A}-\text{BC})}$,

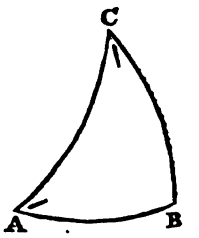
and N = ar. co. log. n.

515. $\left\{\begin{array}{l}\text{TanAC} = \frac{1}{n}.\text{sinBC.} \\ \text{TanAB} = \frac{1}{n}.\text{sinBC.cosA.} \\ \text{Tan C} = \frac{1}{n}.\text{cosA.}\end{array}\right.$

	9.9965299	N	$0.3398449\frac{1}{2}$
	9.3237802	SinBC..	9.7618208
	2)19.3203101	TanAC	$10.1016657\frac{1}{2}$
		Cos A	9.8299589
n..	$9.6601550\frac{1}{2}$		
N	$0.3398449\frac{1}{2}$	TanAB	$9.9316246\frac{1}{2}$
		Tan C	$10.1698038\frac{1}{2}$

$\left.\begin{array}{l}\text{AC} = 51^\circ\ 38'\ 45''.15 \\ \text{AB} = 40^\circ\ 30'\ 29''.50 \\ \text{C} = 55^\circ\ 55'\ 33''.73\end{array}\right\}$

CASE VI. Given the angle A, 41° 54′, and the angle C, 63° 47′.



1. To find one of the sides, as AB.

516. $\left\{\begin{array}{c}\text{Cos AB} = \text{cosec A} \cdot \text{cos C.} \\ \text{AB, C like.}\end{array}\right\}$ Cosec A 0.1753324
Cos C 9.6451931

AB = 48° 35′ 10″.08 Cos AB 9.8205255

Or thus:

517. $\left\{\begin{array}{l}\text{Tan}^2\tfrac{1}{2}\text{AB} = \cot\tfrac{1}{2}(\text{A} + {}^{c}\text{C}). \\ \qquad\qquad \times \tan\tfrac{1}{2}(\text{A} \sim {}^{c}\text{C}).\end{array}\right\}$ 34° 3′ ½cot 10.1700592
7° 51′ ½tan 9.1390093

${}^{c}\text{C}$ = complement of C.

½AB = 24° 17′ 35″.04 2)19.3090685

AB = 48° 35′ 10″.08 Tan½AB $9.6545342\frac{1}{2}$

2. To find the hypotenuse AC.

518. $\left\{\begin{array}{l}\text{Cos AC} = \cot \text{A} \cdot \cot \text{C.} \\ \text{If A, C like, AC} \lessdot 90°, \text{ if not, } >.\end{array}\right\}$ CotA 10.0470872
CotC 9.6923378

AC = 56° 42′ 51″.59 CosAC 9.7394250

Or thus:

519. $\left\{\begin{array}{r}\text{Tan}^2\tfrac{1}{2}AC=\sec(A-C)\\ \times\cos(A+C).\end{array}\right\}$ $\begin{array}{l}21^\circ\,53'\ \sec\ 0.0324779\\ 105^\circ\,41'\ \cos\ 9.4318788\end{array}$

$\tfrac{1}{2}AC=28^\circ\,21'\,25''.80$ $\qquad$ $2)19.4643567$

$AC=56^\circ\,42'\,51''.6$ $\qquad$ $\text{Tan}\tfrac{1}{2}AC\ \ 9.7321783\tfrac{1}{2}$

3. To find all the three sides.

Put $N=\sqrt{\cos(A+C)\,.\,\cos(A-C)}$.

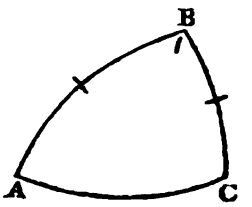
520. $\left\{\begin{array}{l}\text{Tan}AB=N.\sec C.\\ \text{Tan}AC=N.\sec C.\sec A.\\ \text{Tan}BC=N.\sec A.\end{array}\right.$

	9.4318788	N	9.6997004½
	9.9675221	Sec C	0.3548069
2)	19.3994009	TanAB	10.0545073½
N	9.6997004½	Sec A	0.1282452
		TanAC	10.1827525½
		TanBC	9.8279456½

$\left.\begin{array}{l}AB=48^\circ\,35'\,10''.08\\ AC=56^\circ\,42'\,51''.60\\ BC=33^\circ\,56'\,10''.71\end{array}\right\}$

PRACTICAL SOLUTION OF THE CASES OF AN OBLIQUE-ANGLED SPHERICAL TRIANGLE.

CASE I. Given the side AB, 95° 24′, the side BC, 78° 44′, and the included angle B, 85° 16′.



1. To find the third side AC.

521. $\left\{\begin{array}{l} \text{Sin}\tfrac{1}{2}S : \sin\tfrac{1}{2}D :: \cot\tfrac{1}{2}B : \tan x. \\ \text{Sin}x : \sin\tfrac{1}{2}D :: \cos\tfrac{1}{2}B : \sin\tfrac{1}{2}AC. \\ \quad \text{or,} \\ \text{Cos}x : \sin\tfrac{1}{2}S :: \sin\tfrac{1}{2}B : \sin\tfrac{1}{2}AC. \end{array}\right.$

AB	95° 24′		
BC	78° 44′	B	85° 16′
$\frac{1}{2}S$	87° 4′	$\frac{1}{2}B$	42° 38′
$\frac{1}{2}D$	8° 20′		

Cosec $\frac{1}{2}S$	0.0005694	.....		Sin	9.9994306
Sin $\frac{1}{2}D$	9.1611639	.....	9.1611639		
Cot $\frac{1}{2}B$	10.0359189	Cos	9.8667026	Sin	9.8307837
Tan x	9.1976522	Cosec	0.8076776	Sec	0.0053299
$\frac{1}{2}AC$...43° 13′ 2″.83		Sin	9.8355441	Sin	9.8355442

AC = 86° 26′ 5″.66

Or thus,

522. $$\begin{cases} \text{Cos}\,\frac{1}{2}S : \cos\frac{1}{2}D :: \cot\frac{1}{2}B : \tan y. \\ \text{Sin}\ y : \cos\frac{1}{2}D :: \cos\frac{1}{2}B : \cos\frac{1}{2}AC. \\ \quad \text{or,} \\ \text{Cos}\ y : \cos\frac{1}{2}S :: \sin\frac{1}{2}B : \cos\frac{1}{2}AC. \end{cases} \quad \begin{array}{l} \frac{1}{2}S = 87^\circ\ 4' \\ \frac{1}{2}D = 8^\circ\ 20' \\ \hline \frac{1}{2}B = 42^\circ\ 38' \end{array}$$

$\text{Sec}\frac{1}{2}S$	1.2909510	.....		Cos	8.7090490
$\text{Cos}\frac{1}{2}D$	9.9953902	.....	9.9953902		
$\text{Cot}\frac{1}{2}B$	10.0359189	Cos	9.8667026	Sin	9.8307837
Tan y	11.3222601	Cosec	0.0004918	Sec	1.3227519
$\frac{1}{2}AC$...	$43^\circ\ 13'\ 2''.83$	Cos	9.8625846	Cos	9.8625846

$AC = 86^\circ\ 26'\ 5''.66$

By one of these four methods of computation, the third side may be found, not only with great facility, but with the utmost accuracy, whatever be the magnitudes of the several data. And, since x and y are known to express half the difference and half the sum of the angles at the base, it is easy to choose that method which is best adapted for the accurate solution of a given example. It is, therefore, only by way of illustrating the preceding theory that a few other methods are subjoined.

Otherwise, [see 364.]

523. $\left.\begin{array}{l} \\ \text{Cases I. II.} \quad R^2 : \sec^2 x \\ \text{III. IV.} \quad R \cdot \sec 2x : \sec^2 x \end{array}\right\}$ $\left.\begin{array}{l} R : \cos B :: \text{Tan}AB.\tan BC : \tan^2 x \\ :: \text{Cos}AB.\cos BC : R.\cos^2 \end{array}\right.$

Cos B	8.9165504		
Tan AB	11.0244403	... Cos	8.9736280
Tan BC	10.7006784	... Cos	9.2908704
	2)20.6416691		
Tan x ...	$10.3208345\frac{1}{2}$	... 2Sec	0.7309410
$2x$ =	128° 55′ 52″.82	Cos	9.7982283
AC =	86° 26′ 5″.67	Cos	8.7936677

Otherwise,

524. $\left.\begin{array}{l} \text{Cos}B : \cot AB :: R : \tan Z. \\ \text{Sin}Z : \sin(Z+BC) :: \cos AB : \cos AC. \end{array}\right\}$ $\left.\begin{array}{l} \text{If B, AB unlike Z} > 90°. \\ \text{If}(Z+BC), \text{AB unlike AC} > \end{array}\right.$

Sec B	1.0834496		
Cot AB	8.9755597 ...	Cos	8.9736280
Tan z	10.0590093 ...	Cosec	0.1230086
z = 131° 7′ 9″.91	209° 51′ 9″.91	sin	9.6970311
BC = 78° 44′			
AC =	86° 26′ 5″.67	cos	8.7936677

Otherwise,

525. $\left\{\begin{array}{c}\text{Sin}\frac{1}{2}\text{D}:\sin\frac{1}{2}\text{B}::\sqrt{\sin\text{AB}.\sin\text{BC}}:\tan m\\ \text{R}:\sec m::\sin\frac{1}{2}\text{D}:\sin\frac{1}{2}\text{AC}.\end{array}\right\}$ [299, 301.]

Sin AB	9.9980683	Cosec ½D	0.8388361	Sin 9.1611639
Sin BC	9.9915488	Sin ½B	9.8307837	
2)19	9896171	.....	9.9948085½	
		Tan	10.6644283½	Sec 0.6743803
AC = 86° 26′ 5″.66		½AC 43° 13′ 2″.83		Sin 9.8355442
&c.			&c.	

2. To find one of the angles, as A.

526. $\left\{\begin{array}{l}\text{SinS}:\sin\text{D}::\cot\frac{1}{2}\text{B}:\tan x.\\ \text{R}:\cot(x+\frac{1}{2}\text{B})::\sec\text{AB}:\tan\text{A}.\end{array}\right\}$ If AB $\angle$ BC, take $\cot(x-\frac{1}{2}\text{B})$. If AB $\angle$ 90°, A&$(x\pm\frac{1}{2}\text{B})$ like.

Cosec S	0.9904904 ...	Sec AB	1.0263720
Sin D	9.4575840 ...	Cot $(x+\frac{1}{2}\text{B})$	9.6580741
Cot ½B	10.0359189 ...	Tan A	10.6844461
Tan x	10.4839933		

$x = 71°\ 50'\ 7''.15$ ..... $\text{A} = 78°\ 18'\ 57''.05$

Or thus,

527. $\left\{\begin{array}{l}\text{TanBC}:\tan\text{AB}::\text{R}.\sec\text{B}:\tan^2 x.\\ \left.\begin{array}{l}\sec^2 x:\text{R}\ .\ \sec\text{AB}\\ \sec^2 x:\sec 2x.\sec\text{AB}\end{array}\right\}::\tan\text{B}:\tan\text{A}\end{array}\right\}$ see 364.

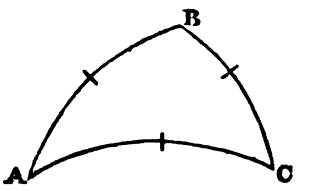
Cot BC	9.2993216		
Tan AB	11.0244403	... Sec	1.0263720
Sec B	1.0834496	... Tan	11.0819660
	2)21.4072115		
Tan x	10.7036057½	... 2Cos	18.5761080
A=78° 18′ 57″.04		Tan	10.6844460
&c.	&c.	&c.	

3. To find both the angles A, C.

528. $\left.\begin{array}{l}\text{Sin}\frac{1}{2}\text{S}:\sin\frac{1}{2}\text{D}::\cot\frac{1}{2}\text{B}:\tan\frac{1}{2}(\text{A}-\text{C}).\\ \text{Cos}\frac{1}{2}\text{S}:\cos\frac{1}{2}\text{D}::\cot\frac{1}{2}\text{B}:\tan\frac{1}{2}(\text{A}+\text{C}).\end{array}\right\}\ \frac{1}{2}(\text{A}+\text{C}),\ \frac{1}{2}\text{S}$ li

Cosec ½S	0.0005694	..... Sec	1.2909510	½Diff.	8° 57′ 29″.15
Sin ½D	9.1611639	..... Cos	9.9953902	½Sum	87° 16′ 26″.20
Cot ½B	10.0359189	.....	10.0359189	C =	96° 18′ 55″.35
Tan ½Diff.	9.1976522	Tan ½Sum	11.3222601	A =	78° 18′ 57″.05

Case II. Given the three sides AB, 67° 38′, BC, 56° 14′, and AC, 83° 20′.



1. To find one of the angles, as B.

529. Sin AB . sin BC : sin ½(AC+D) . sin ½(AC—D) : : R^2 : sin ^{2} ½B.

AB 67° 38′	.....	Cosec	0.0339674
BC 56° 14′	.....	Cosec	0.0802381
D 11° 24′	½Sum 47° 22′	Sin	9.8667026
AC 83° 20′	½Diff. 35° 58′	Sin	9.7688707
			2)19.7497788
B=97°7′45″.70	½B=48°33′52″.85	Sin	9.8748894

Or thus,

530. Sin AB . sin BC : sin ½(S+AC) . sin ½(S—AC) : : R² : cos² ½B.

AB 67° 38′	.....	Cosec	0.0339674
BC 56° 14′	.....	Cosec	0.0802381
S 123° 52′	½Sum 103° 36′	Sin	9.9876488
AC 83° 20′	½Diff. 20° 16′	Sin	9.5395653
			2)19.6414196
B=97° 7′ 45″.70	48° 33′ 52″.85	Cos	9.8207098

Or thus,

531. Sin P. sin(P—AC) : sin(P—BC) . sin(P—AB) : : R² : tan² ½B.

AB 67° 38′	P	103° 36′	Cosec	0.0123512
BC 56° 14′	P—AC	20° 16′	Cosec	0.4604347
AC 83° 20′	P—BC	47° 22′	Sin	9.8667026
2)207° 12′	P—AB	35° 58′	Sin	9.7688707
P 103° 36′		207° 12′	2)20.1083592	
B=97° 7′ 45″.70	48° 33′ 52″.85		Tan	10.0541796

Or thus,

532. $\left\{\begin{array}{l} R : \cos AC :: \sec AB : \sec BC : \tan^2 x. \\ R^2 : \sec^2 x \\ R\sec 2x : \sec^2 x \end{array}\right\} :: \cot AB . \cot BC : R . \cos B \Big\}$ see 364.

Cos AC	9.0648057		
Sec AB	0.4196083 ...	Cot	9.6143591
Sec BC	10.2550720 ...	Cot	9.8251660
	2)19.7394860		
36° 32′ 1″.32	Tan 9.8697430 ...	2Sec	0.1900210
73° 4′ 2″.64	.....	Cos	6.4642608
B = 97° 7′ 45″.71	...	Cos	9.0938069

Or thus,

533. $\left\{\begin{array}{l} \text{Sin } \frac{1}{2}AC : \sin \frac{1}{2}D :: R : \sin m. \\ \sqrt{\sin AB \cdot \sin BC} : \cot m :: \sin \frac{1}{2}D : \sin \frac{1}{2}B. \end{array}\right\}$

SinAB	9.9660326	Cosec ½AC	0.1773117		
SinBC	9.9197619	Sin ½D	8.9970356		8.9970356
	2)19.8857945		9.1743473	Cot	10.8207511
	9.9428972½			ar. co.	0.0571027½
B = 97° 7′ 45″.70	...	48° 33′ 52″.85		Sin	9.8748894½
&c.				&c.	

2. To find two of the angles, as A, C.

534. $\left\{\begin{array}{l} \text{Tan } \frac{1}{2}S : \tan \frac{1}{2}AC :: R : \tan x. \\ \text{Tan } \frac{1}{2}AC : \tan \frac{1}{2}D :: R : \tan y. \end{array}\right.$ $\left|\begin{array}{l} \text{Tan}^2 \frac{1}{2}A = \cot(45+x).\cot(45+y). \\ \text{Tan}^2 \frac{1}{2}C = \cot(45+x).\tan(45+y). \end{array}\right.$

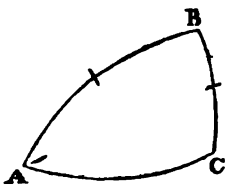
	Tan½AC	9.9493531	.....	Cot	10.0506469
	Cot ½S	9.7368925	.....	Tan½D	8.9991883
x=25° 23′ 5″.35	Tan	9.6762456	y=6° 23′ 58″.23	Tan	9.0498352
45°					
70° 23′ 5″.35	Cot	9.5519165	.....		9.5519165
51° 23′ 58″.23	Cot	9.9021680	.....	Tan	10.0978320
		2)19.4540845			2)19.6497485
½A=28° 4′ 29″.54	Tan	9.7270422½		Tan	9.8248742½
A=56° 8′ 59″.08			C=67° 29′ 51″.94		

3. To find all the angles.

535.
$$\begin{cases} \text{SinP.sin(P–AC)} : \text{sin(P–AB).sin(P–BC)} :: R^2 : \tan^2\tfrac{1}{2}B. \\ \text{SinP} : \text{sin(P–AB)} :: \cot\tfrac{1}{2}B : \tan\tfrac{1}{2}A. \\ \text{SinP} : \text{sin(P–BC)} :: \cot\tfrac{1}{2}B : \tan\tfrac{1}{2}C. \end{cases}$$

AB 67° 38′	Cosec(P—AC)	0.4604347		
BC 56° 14′	Cosec P	0.0123512	0.0123512	0.0123512
AC 83° 20′	Sin (P—AB)	9.7688707	9.7688707	
2)207° 12′	Sin (P—BC)	9.8667026		9.8667026
P 103° 36′		2)20.1083592		
		Tan 10.0541796	Cot 9.9458204	9.9458204
			Tan 9.7270423	Tan 9.8248742
		48° 33′ 52″.85	28° 4′ 29″.55	33° 44′ 55″.96
		B = 97° 7′ 45″.70.	A = 56° 8′ 59″.10.	C = 67° 29′ 51″.92

CASE III. Given the side AB, 65° 34′, the side BC, 41° 52′, and the angle A opposite to BC, 46° 35′.



1. To find the angle C opposite to AB.

536. $\left\{\begin{array}{l}\text{Sin BC : sin AB : : sin A : sin C.}\\ \dfrac{A+C}{2} \text{ like } \dfrac{AB+BC}{2}.\end{array}\right.$

Cosec BC	0.1756142		
Sin AB	9.9592528		
Sin A	9.8611608	or C =	97° 44′ 14″.83
Sin C	9.9960278	C =	82° 15′ 45″.17
			180°

2. To find thence the other angle B, and the remaining side AC.

537. $\left\{\begin{array}{l}\text{Sin } \frac{1}{2}S : \sin \frac{1}{2}D : : \cot \frac{1}{2}(A - C) : \tan \frac{1}{2}B.\\ \text{Sin } \frac{1}{2}(A-C) : \sin \frac{1}{2}(A+C) : : \tan \frac{1}{2}D : \tan \frac{1}{2}AC.\end{array}\right\}$

C	82° 15′ 45″.17	97° 44′ 14″.83	Cosec ½S	0.0996108
A	46° 35′	46° 35′	Sin ½D	9.3124951
	2)35° 40′ 45″.17	51° 9′ 14″.83		9.4061059
	17° 50′ 22″.58	25° 34′ 37″.42	Cot	10.3200023
Cot	10.4923768		Tan	9.7261082
	9.4061059			28° 1′ 25″.43
Tan	9.8984827	38° 21′ 49″.16		2
		2		B = 56° 2′ 50″.86
		B = 76° 43′ 38″.32		

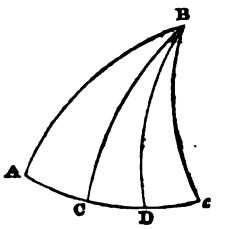
C 82° 15′ 45″.17	½Sum	64° 25′ 22″.58	Sin	9.9552092
A 46° 35′	½Diff.	17° 50′ 22″.58	Cosec	0.5137773
	½D	11° 51′	Tan	9.3218506
AC = 63° 24′ 51″.40		31° 42′ 25″.70	Tan	9.7908371

or,

C 97° 44′ 14″.83	½Sum	72° 9′ 37″.42	Sin	9.9785994
A 46° 35′	½Diff.	25° 34′ 37″.42	Cosec	0.3647932
	½D	11° 51′	Tan	9.3218506
AC = 49° 39′ 15″.48		24° 49′ 37″.74	Tan	9.6652432

Every oblique-angled triangle may be resolved into two right-angled triangles, from which its parts may also be computed by the theorems respecting the latter.

In this last case, for example, let BD be perpendicular to AC.



1. To find the angle C.

538.
$$\begin{cases} \text{Sin BD} = \sin \text{AB} \cdot \sin \text{A} \\ \text{Tan}^2 \tfrac{1}{2} \text{C}^c = \cot \dfrac{\text{BC}+\text{BD}}{2} \cdot \tan \dfrac{\text{BC}-\text{BD}}{2}. \end{cases}$$

C^c = complement of C.

Sin BD 9.8204136	BD = 41° 24′ 3″.05	
See above.	BC = 41° 52′	
	½Sum 41° 38′ 1″.52	Cot 10.0511493
	½Diff. 00° 13′ 58″.47	Tan 7.6090647
		2)17.6602140
	½C^c 3° 52′ 7″.42	8.8301070
	2	
	C = 97° 44′ 14″.84	
	or 82° 15′ 45″.16	

2. To find the other side AC.

539. $\begin{cases} \text{Tan}^2\tfrac{1}{2}AD = \tan\tfrac{1}{2}(AB+BD).\tan\tfrac{1}{2}(AB-BD). \\ \text{Tan}^2\tfrac{1}{2}DC = \tan\tfrac{1}{2}(BC+BD).\tan\tfrac{1}{2}(BC-BD). \end{cases}$

½(AB+BD) = 53° 29′ 1″.52	tan 10.1305336	½sum (above)	tan 9.94885
½(AB−BD) = 12° 4′ 58″.48	tan 9.3305548	½diff.	tan 7.60906
	2)19.4610884		2)17.55791
AD = 56° 32′ 3″.48	Tan 9.7305442		Tan ½DC 8.77895
DC = 6° 52′ 47″.98			
AC = 63° 24′ 51″.46			
AC = 49° 39′ 15″.50			

3. To find the other angle B.

540. $\begin{cases} \text{Tan ABD} = \sec AB \,.\, \cot A. \\ \text{Tan CBD} = \sec BC \,.\, \cot C. \end{cases}$

AB sec 0.3833836	BC sec 0.1280187	ABD = 66° 23′ 14″.67
A cot 9.9759849	C cot 9.1331269	CBD = 10° 20′ 23″.73
ABD tan 10.3593685	CBD tan 9.2611456	ABc = 76° 43′ 38″.40
		ABC = 56° 2′ 50″.94

OF THE ACCURACY OF LOGARITHMIC CALCULATIONS.

541. As the logarithmic tables consist almost wholly of approximate decimals, the last figure in each logarithm must generally be too great or too small. The greatest error arising from thence is $\frac{1}{2}$ in the last figure. By adding two logarithms together, therefore, the error in the last figure of the sum may be 1; by adding three, the error may be $1\frac{1}{2}$; and so on.

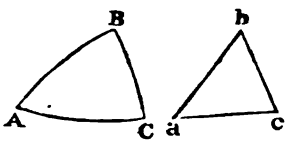
The greatest possible value of 1 in the 7th decimal of a logarithmic tangent is 0″.024; and that is when the arc is near to 45°. But in general its value is too small to be regarded.

Whereas, in logarithmic sines, the value of 1 in the last figure of the sine 45° is 0″.048, (double of that in the tangent), of the sine $64°\frac{1}{2}$ is 0″.1, of the sine $84°\frac{1}{2}$ it is 0″.5, of the sine 87° it is a whole second, of the sine $88°\frac{1}{2}$ it is 2″, &c. &c.

Hence we may conclude, that any inaccuracy in the results of trigonometrical calculations may be chiefly

attributed to the finding of large arcs from their sines, or small ones from their cosines. It is better, therefore, to avoid using sines above 70°, when the calculation can be effected by tangents.

EXEMPLIFICATION OF LEGENDRE'S THEOREM.



In the spherical triangle ABC, given the side AB=58′, BC = 44′, and the included angle B=73° 14′ 20″, to find the angles of the plane rectilineal triangle *abc*, the corresponding sides of both triangles being of the same length.

$$\left\{\begin{array}{l}\text{Sin } 51' : \sin 7' :: \cot \tfrac{1}{2}B : \tan \dfrac{A-C}{2}. \\ \\ \text{Cos } 51' : \cos 7' :: \cot \tfrac{1}{2}B : \tan \dfrac{A+C}{2}.\end{array}\right\} \text{see } 528.$$

Cosec 51′	1.8287196	Sec 51′	0.0000478	½Sum	53° 23′ 00″.659
Sin 7′	7.3088239	Cos 7′	9.9999991	½Diff.	10° 27′ 50″.466
Cot ½B	10.1288988		10.1288988	C =	63° 50′ 51″.125
				A =	42° 55′ 10″.193
Tan	9.2664423	Tan	10.1289457	B =	73° 14′ 20″.

Sum of the 3 angles of the sph. triangle = 180° 00′ 21″.318

⅓ of the excess above 180° is 7″.106, which being sub-

tracted from each of the said angles, we get the angles of the plane triangle, viz.

$a=42°55'3''.087$, $b=73°14'12''.894$, $c=63°50'44''.019$.

And from either of these triangles the base may be found $=$ 61′.86, or 61′ 52″.2.

[redacted]

[redacted] of either

[redacted]

[redacted], or pas-

[redacted] ecliptic,

[redacted], or circles

[redacted]

[redacted] is the equator,

[redacted] obliquity; and

[redacted] called the equi-

[redacted] divided into twelve

[redacted] first sign begins

[redacted] their names and cha-

		Lat.	Long.
[redacted]	[redacted]	1	2
[redacted]	[redacted]	0	0

[redacted] Aspects. Planets.

[redacted] 2 [redacted], the other six

[redacted] 2 [redacted] the equator nearest

[redacted] 2 [redacted], the other pole the

[redacted]

[redacted] equal great circles per-

[redacted] of the ecliptic of any

[redacted] 2 [redacted] in either reckoning

[redacted] 2 [redacted] 1 [redacted] 1 [redacted] of longitude

[redacted]

ASTRONOMICAL PROBLEMS.

DEFINITIONS.

[From the blank which occurs in this part of the manuscript, it seems probable that the Author had intended some introduction to the following Astronomical Problems. To supply the defect, a few explanatory remarks are here inserted under the form of definitions.

These problems concern chiefly the finding, from proper data, the measures of certain arcs and angles formed on the surface of the celestial and terrestrial spheres, [221, &c.] by the apparent motions and positions of the celestial bodies. The spheres are supposed to be concentric with the earth, and to have corresponding circles described on them.

The *axis* of the earth is the straight line about which it turns; and being produced, becomes the *celestial axis*, or *axis of the world*, about which the heavens seem to revolve.

The *equator* or *equinoctial* is a great circle of either sphere, having its poles in the axis of the world.

Great circles perpendicular to the equator, or passing through the axis of the world, are called *meridians*, or *hour circles*, or *circles of right ascension*, or *circles of terrestrial longitude*.

The *ecliptic* is a great circle inclined to the equator, at an angle of about 23° 28′, called its *obliquity*; and the intersections of these two circles are called the *equinoctial points*. The ecliptic is divided into twelve parts of 30° each, called *signs*. The first sign begins at one of the equinoctial points. Their names and characters are,

Aries,	*Taurus,*	*Gemini,*	*Cancer,*	*Leo,*	*Virgo,*
♈	♉	♊	♋	♌	♍
Libra,	*Scorpio,*	*Sagittarius,*	*Capricornus,*	*Aquarius,*	*Pisces.*
♎	♏	♐	♑	♒	♓

The first six are called the *northern*, the other six the *southern* signs. The pole of the equator nearest the former is named the *north* pole, the other pole the *south*.

Circles of celestial longitude are great circles perpendicular to the ecliptic; and the *longitude* of any point in the heavens is the arc of the ecliptic reaching from the first point of Aries, to a circle of longitude passing through that point.

The *latitude* of any point in the heavens is the intercepted arc of a great circle perpendicular to the ecliptic, and is called *north* or *south* latitude, according as the point lies north or south of the ecliptic.

An arc of the equator, reaching from the first point of Aries to a meridian passing through the centre of an object, is called its *right ascension.*

An arc of a meridian, intercepted between the centre of an object and the equator, is termed its *declination,* and is *north* or *south* declination, according as the object lies north or south of the equator.

The *zenith* is that point in the heavens directly over any place; and the *nadir* is the point diametrically opposite.

The *horizon* of any place is the great circle of the sphere equally distant from the zenith and nadir, and which divides the sphere into the upper and lower hemispheres.

Azimuth or *vertical circles* are great circles passing through the zenith and nadir. The *prime vertical* cuts the horizon in the east and west points, and the *meridian* of the place cuts the horizon perpendicularly in the north and south points.

The *azimuth* of an object is an angle at the zenith, formed by the meridian of the place, and the vertical or azimuth circle passing through the object.

Altitude is the arc of a vertical circle reaching from the object to the horizon, and *zenith distance* is the complement of that arc.

Longitude of a place on the earth is the arc of the equator intercepted between the meridian of the place and a given meridian, called the *first* meridian.

Latitude on the earth is an arc of the meridian between the place and the equator, and is called *north* or *south* latitude, according as the place lies north or south of the equator. This arc corresponds to declination in the heavens, and the complement of either is called the *polar distance*.

Parallax is the difference of the place of an object as seen from the surface and from the centre of the earth, and is distinguished into *parallax in altitude*, in *longitude*, *latitude*, &c., according as it affects the apparent magnitude of these arcs.

Refraction of light is the bending which rays suffer in passing through media of different density or composition. It is called *refraction in altitude*, *on distance*, &c., according as it affects the apparent lengths of such arcs. Both atmospherical refraction and parallax are, generally speaking, greatest when the object appears at the earth's surface, and they gradually diminish and become extinct as it approaches the ze-

nith. Parallax reduces the apparent altitude of bodies, while the atmospherical refraction augments it.

Aberration is an apparent change in the position of celestial bodies, occasioned by the progressive motion of their light combined with the earth's annual motion. It is well illustrated in Problems 20. and 21.

Precession of the equinoxes is a slow retrograde motion of the equinoctial points at the annual rate of about 50″. It is caused by the joint action of the sun and moon on the protuberant matter of the earth's equatorial regions.

Nutation, a tremulous motion of the earth's axis which slightly disturbs the uniformity of the precession, the obliquity of the ecliptic, and the places of the stars. It is caused by the variable attraction of the moon, and has a period of about 18.6 years.

The *orbit* of a planet is the curvilinear path in which it moves.

Apses or *apsides* are the two points in the orbit of a planet, which are nearest and farthest from its primary. They are called *perihelion* and *aphelion* in regard to the sun, but *perigee* and *apogee* when referred to the earth. A straight line joining these opposite points is called the *line of the apsides*.

Conjunction occurs when two heavenly bodies, seen from the sun or earth, have the same longitude.

When one heavenly body is hid behind or in the shadow of another, it is said to be *eclipsed*. But this phenomenon is frequently called an *occultation*, in the case of a star, or other apparently small body.

The interposition of the moon between the sun and the earth causes a solar eclipse; and the intervention of the earth between the sun and the moon, to which it extends its dark shadow, produces a lunar eclipse. Eclipses are *total* or *partial*, according as the whole or part only of the luminary is obscured.

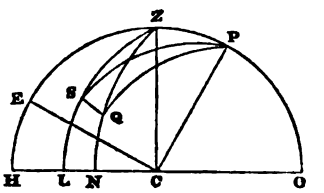
Phases are the different appearances of the moon, planets, &c. during an eclipse, or other remarkable phenomenon.]

ASTRONOMICAL PROBLEMS.

PROBLEM I.

Given two altitudes of the sun, and the times from noon, to find the latitude of the place, and the sun's declination, supposing it invariable during the intermediate time.

Let Z be the zenith of the place, HO the horizon, EC the equator, and P its pole ; SL, QN the sun's altitudes at the times of observation, SP, QP his polar distances (supposed to be equal,) and EZ the latitude.



Put A, a for the two altitudes, T, t for the times from noon (or measures of the angles QPZ, SPZ), L, D for the latitude and declination, Ss for the rectangle of their sines, and Cc for the rectangle of their cosines.

Then, from the triangles ZPS, ZPQ we have (Rad = 1),

$$\left.\begin{aligned} \text{Cos } t \ . \ Cc + Ss &= \sin A. \\ \text{Cos T} \ . \ Cc + Ss &= \sin a. \end{aligned}\right\} [266.]$$

$$\text{Hence, } Cc = \frac{\sin A - \sin a}{\cos t - \cos T} = M \text{ suppose.}$$

$$\text{And } 2Ss = \sin A + \sin a - M \ . \ (\cos t + \cos T).$$

$$\text{But } \sin A - \sin a = M \ . \cos t - M \ . \cos T.$$

$$\text{Therefore } Ss = \sin A - M \ . \cos t = N \text{ suppose.}$$

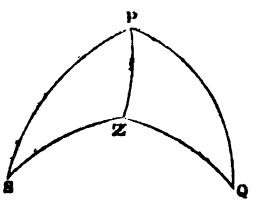
$$\left.\begin{aligned} \text{Consequently } \cos (L + D) &= M \backsim N. \\ \cos L \backsim D &= M + N. \end{aligned}\right\} [31, 32.]$$

When the latitude and declination are on different sides of the equator, and t less than 6^h or $90°$, it is $M \ . \cos t - \sin A = N$. When they are on the same side, and both times greater than 6^h, it is $\sin A + M \ . \cos t = N$. [363.]

PROBLEM II.

To find the equation of time, for determining apparent noon, from equal altitudes of the sun.

Let Z be zenith, P the elevated pole, and S the sun's centre. The latitude L, declination D, and time from noon SPZ ($= t$) being given; it is required to determine the effect of a small change d in the declination upon the time t, while the altitude (or the zenith distance ZS) remains the same.



In the triangle PZS, because PZ, ZS are invariable, $PS'' : P'' :: \sin PS : \cot S$ [410.] $:: \sin PS : \frac{\sin PS . \cot PZ}{\sin P} -$ $\cos PS . \cot P$ [270.] $:: 1 : \frac{\cot PZ}{\sin P} - \cot PS . \cot P$. Hence, $1 : \frac{\tan L}{\sin t} - \frac{\tan D}{\tan t} :: d : x$, the quantity required. But when the declination and latitude are of a different name, $\frac{\tan D}{\tan t}$ becomes affirmative.

Thus, if there be two equal altitudes (at S in the forenoon, and Q in the afternoon,) the intermediate time being $2t$, and $SP = (PQ + d)$; then $SPZ = t$ nearly, and $x = \frac{d \tan L}{\sin t} \mp \frac{d \tan D}{\tan t}$ is the quantity, which, subtracted from t (because the polar distance is decreasing), gives the value of the angle SPZ very nearly.

Otherwise,

The angles SPZ, QPZ being expressed by $(t - \frac{1}{2}x)$ and $(t + \frac{1}{2}x)$, we have $\cos (t - \frac{1}{2}x) . \sin PS . \sin PZ + \cos PS . \cos PZ = \cos SZ$ [266.] $= \cos (t + \frac{1}{2}x) . \sin PQ . \sin PZ + \cos PQ . \cos PZ$; therefore $\cos (t - \frac{1}{2}x) . \sin PS - \cos (t + \frac{1}{2}x) . \sin PQ = \cot PZ . (\cos PQ - \cos PS)$.

But $\cos (t - \frac{1}{2}x) = \cos t . \cos \frac{1}{2}x + \sin t . \sin \frac{1}{2}x$. [32.]

$\cos t + \frac{1}{2}x = \cos t . \cos \frac{1}{2}x - \sin t . \sin \frac{1}{2}x$. [31.]

Hence $\sin t \,.\, \sin \frac{1}{2}x\,(\sin \mathrm{PS} + \sin \mathrm{PQ}) + \cos t \,.\, \cos \frac{1}{2}x\,(\sin \mathrm{PS} - \sin \mathrm{PQ}) = \tan \mathrm{L}\,(\cos \mathrm{PQ} - \cos \mathrm{PS})$. And since the sum of the sines of two arcs, their difference, and the difference of their cosines, are proportional to the cotangent of half their difference, the cotangent of half their sum, and the radius [43, 45.], $\cot \frac{1}{2}d \,.\, \sin t \,.\, \sin \frac{1}{2}x + \cot (\mathrm{PQ} + \frac{1}{2}d) \,.\, \cos t \,.\, \cos \frac{1}{2}x = \tan \mathrm{L}$. Hence, putting $(\mathrm{PQ} + \frac{1}{2}d) = \mathrm{D}$, the mean declination, or the declination at noon, we find

$$\mathrm{Tan}\ \tfrac{1}{2}x = \frac{\tan \frac{1}{2}d \,.\, \tan \mathrm{L} \,.\, \sec \frac{1}{2}x}{\sin t} - \frac{\tan \frac{1}{2}d \,.\, \tan \mathrm{D}}{\tan t}.$$

of which the last term becomes affirmative, when the sun's polar distance is above 90°.

Or thus,

The equation $\cot \frac{1}{2}d \,.\, \sin t \,.\, \sin \frac{1}{2}x + \cot \mathrm{D} \,.\, \cos t \,.\, \cos \frac{1}{2}x = \tan \mathrm{L}$, may be accurately resolved in this manner:

$$\mathrm{Tan}\ z = \cot t \,.\, \tan \mathrm{D} \,.\, \tan \tfrac{1}{2}d.$$

$$\mathrm{Sin}\ y = \sec t \,.\, \cot \mathrm{D} \,.\, \tan \mathrm{L} \,.\, \sin z.$$

Then $\frac{1}{2}x = y \mp z$. — if $\mathrm{D} \lessgtr 90°$, + if $>$,

by means of the following useful

LEMMA.

Given the equation $a \cos x + b \sin x = c$, to find x.

Suppose $c = n \,.\, \sin (y + x)$, then

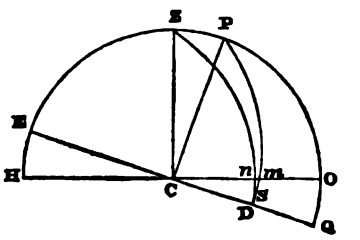
$a \cos x + b \sin x = n \,.\, \sin y \,.\, \cos x + n \,.\, \cos y \,.\, \sin x$. [29.]

By equating the corresponding terms of this equation, we get $\frac{a}{\sin y} = n = \frac{b}{\cos y}$. Hence tan $y = \frac{a}{b}$, and $\sin(y+x) = \frac{c}{n} = \frac{c.\sin y}{a}$, or $\sin(y+x) = \frac{c.\cos y}{b}$. Thus, both y and $(y + x)$, and consequently x itself, are given.

PROBLEM III.

To find the apparent time of the rising or setting of the sun's upper limb, at a given place, the horizontal refraction being given.

Let Z be the zenith, HO the horizon, &c. When the sun's upper limb appears at n in the horizon, as it is elevated about 33′ by refraction, and its semidiameter is 16′, the true place of its centre will be 49′ below the horizon: And, to an observer elevated above the horizon, it is still farther depressed. Thus, if S be the centre, the arc Sn in the vertical circle ZnS is equal to the sum of the horizontal refraction, the semidiameter, and the dip of the ho-



rizon, $= s$ suppose. Let the hour circle PSD through S meet the horizon in m, and the equator ECQ in D. It is required to find the arc CD, which measures the time from 6^h.

The sun's declination SD, being very nearly the same with the declination (D) at 6^h, is given, which being increased or diminished by $Sn = s$, according as the latitude and declination are on the same or on different sides of the equator, gives mD nearly; to find it more nearly, we have [302, 301.]

$$\text{Cos } m\text{ CD} : \cos m\text{D} (:: \sin m : \text{R}) :: \sin \text{S}n : \sin \text{S}m ;$$

$$\text{that is, } \sin \text{L} : \cos (\text{D} \pm s) :: \sin s : \sin s' (\text{S}m) ;$$

$$\text{then } m\text{D} = \text{D} \pm s', \text{ and [301.]}$$

$$\sin \text{CD} = (\cot m\text{CD} . \tan m\text{D} =) \tan \text{L} . \tan (\text{D} \pm s').$$

Having thus found nearly the time from 6^h, the declination corresponding to that time may be substituted in place of D in the last equation, and CD more accurately determined, if necessary.

Otherwise,

Taking the declination at the preceding noon of Greenwich ($=$ D), as the value of SD, we have all the sides of the triangle ZPS given, to find the angle at P [521, 522.], by which the apparent time at the given place, and thence the apparent time at Green-

wich, is known nearly. Find then the increment or decrement of declination from the preceding noon to that time ($= D''$), and the corresponding increment or decrement of the time CD ($= t''$) may be found thus,

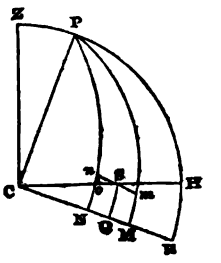
$$1 : \frac{\tan L}{\sin t} = \frac{\tan D}{\tan t} :: D'' : t'',$$

as appears from the solution of the preceding problem.

PROBLEM IV.

To find the latitude, from the sun's declination and diameter, and the time of rising or setting of the whole disc; also, to find the time, when the latitude is given.

Let Z be the zenith, P the pole, S the sun's centre in the horizon CH, SQ his declination, and m, n the apparent places of the centre (on the parallel of declination mSn), when the upper and lower limbs just touch the horizon, no perpendicular to CH, being the semi-diameter. The angle MPN (or arc MN), therefore, measures the time of rising of the whole disc, which, put $= t$, declination $= D$, diameter $= d$, and the angle CSQ $= y$.



Now MN : mn : : MN : $2n$S : : R : cos SQ,

and $2n$S : $2no$: : R : sin nSo(cos CSQ).

Hence $t : d$: : R^2 : cos D . cos y, and from the triangle CQS, R : sin y : : cos D : sin L.

Or thus,

Let $\frac{t}{d} = \cos A$; then, from the last analogy, we have $\cos^2 D - \sin^2 L = \cos^2 D \cdot \cos^2 y = \cos^2 A$, therefore $\sin^2 L = \cos^2 D - \cos^2 A = \sin^2 A - \sin^2 D$. Consequently,

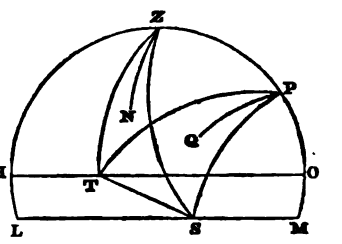
$$\text{Sin L} = \sqrt{\sin (A + D) \cdot \sin (A - D)}. \text{ [38.]}$$

To find the time, when the latitude is given.

We have $\begin{cases} \cos D : \sin L : : R : \sin y. \\ \cos D : \sec y : : d : t. \end{cases}$

PROBLEM V.

To find the day and duration of the shortest twilight at a given place.



Let Z be the zenith of the given place, P the pole, HO the horizon, LM a parallel to HO at the distance of 18°, which is supposed to be the limit of twilight. It is required to find the shortest time of

the sun's moving from one of these circles to the other on a parallel to the equator, as from S to T; that is, to find the angle SPT, when a minimum; and also to find the polar distance PS or PT.

Let PQ, ZN bisect the angles SPT, SZT; and, from the triangle ZPS, which varies to ZPT, while the two sides ZP, PS remain the same, we have these analogies (in which S, T denote the angles at these points subtended by the common side PZ).

$$\text{Sin } \frac{T-S}{2} : \sin TPQ :: \sin PZ.\cos PZN : R.\sin \frac{ZS+ZT}{2}.$$

[411.]

$$\text{Sin } TPQ : \sin \frac{ZS-ZT}{2} :: R.\cos \frac{T-S}{2} : \sin PZ.\sin PZN.$$

[413.]

From the latter analogy it is plain that sin TPQ is a minimum, when $\frac{\cos\frac{1}{2}(T-S)}{\sin PZN}$ is a minimum (the other terms being given,) that is, since the difference of T, S must be very small in respect to the variation of PZS, when sin PZN is a maximum, or equal to the radius; in which case cos PZN $= o$. In the former analogy, therefore, sin $\frac{T-S}{2} = o$, or T $=$ S. Hence the second analogy becomes,

$$\text{Sin } TPQ : \sin 9^\circ :: R : \cos \text{ lat.},$$

which determines the angle TPQ, or the measure of $\frac{1}{2}$ time required.

Again, since the angles T, S are equal, either of them is half the difference of the angles ZTS, ZST. Hence,

$\text{Sin}\frac{1}{2}(ZS+ZT):\sin\frac{1}{2}(ZS-ZT)::\cot TZN:\tan T$. [284.]

or, $\cos 9^\circ : \sin 9^\circ : : \tan PZT : \tan T$.

But from the quadrantal triangle PZT, we have,

$$\left.\begin{array}{l} R \cdot \cos PZ = \tan PZT \cdot \cot P. \\ R \cdot \cos PT = \tan T \quad \cdot \cot P. \end{array}\right\} \text{[304, 303.]}$$

Whence $\tan PZT : \tan T : : \cos PZ : \cos PT$.

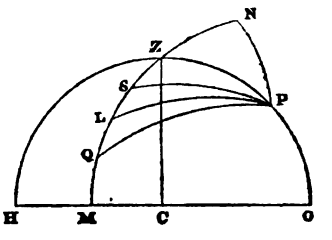
Consequently,

$R : \tan 9^\circ : : (\cos PZ : \cos PT : :) \sin \text{lat.} : \sin \text{decl.}$

Thus, the sun's declination (of a contrary name to the latitude,) and thence the day or days of the year may be obtained.

PROBLEM VI.

To find the zenith's right ascension, at a given place, from two known stars having the same azimuth; also, to find that azimuth.



Let S, Q be the two stars in the same vertical quadrant ZM; the polar distances SP, PQ, the angle SPQ which is the difference of their right ascensions, and PZ the complement of the latitude, are given to find the angle ZPS, whence the right ascension of the zenith will be known.

From the triangle SPQ, supposing PN perpendicular to the base SQ, and PL to bisect the vertical angle, we have,

$$\text{Sin}(SP + PQ) : \sin(SP \smile PQ) :: \cot \tfrac{1}{2}SPQ : \tan LPN.$$

[332.]

If $SP = PQ$, PN and PL coincide; otherwise,

$$LPN + \tfrac{1}{2}SPQ = SPN, \text{ if } PS \succ PQ.$$

$$LPN \smile \tfrac{1}{2}SPQ = SPN, \text{ if } PS \prec PQ.$$

Then, from the triangle ZPS, we have [327.]

$$\tan PZ : \tan PS :: \cos SPN : \cos ZPN,$$

where ZPN is always less than 90°.

Now, if $PS \prec PQ$, and at the same time $LPN \succ \frac{1}{2}SPQ$, the sum, otherwise the difference, of SPN, ZPN is the angle ZPS, which added to or subtracted from the right ascension of the higher star, according as it is on

the west or east side of the meridian, gives the right ascension of the zenith.

To find the azimuth, R : tan ZPN : : sin lat. : cot PZN. Or, the azimuth may be determined independently of the preceding operations, by first finding the angle Q from the triangle SPQ, and then the angle at Z from the triangle PQZ.

Otherwise,

Put SP$=p$, PQ$=q$, the angle SPQ$=d$, and SPZ$=x$, then [348.] $\cot p . \sin(d+x) = \cot q . \sin x + \tan \mathrm{L} . \sin d$, and, [29.]

$(\cot p . \cos d - \cot q) . \sin x + \cot p . \sin d . \cos x = \tan \mathrm{L} . \sin d$.

Hence, (by preceding Lemma),

$$\text{if } \cot y = \frac{\cot p . \cos d - \cot q}{\cot p . \sin d} = \cot d - \frac{\tan p . \cot q}{\sin d},$$

$$\text{then } \sin(y+x) = \frac{\tan \mathrm{L} . \sin d . \sin y}{\cot p . \sin d} = \tan \mathrm{L} . \tan p . \sin y.$$

Or thus, [34.] if $\tan z = \tan q . \cot p . \sin d$,

then, $\cot y = \sin(d \sim z) . \operatorname{cosec} d . \operatorname{cosec} z$.

And, $\sin(y+x) = \tan \mathrm{L} . \tan p . \sin y$.

LEMMA 2.

The difference of the sines of two altitudes of the sun (on the same day and in the same latitude) is pro-

portional to the difference of the versed sines of the times from noon.

For in the triangle ZPS (see figure to Problem VIII.), putting L=latitude, D=declination, A=altitude SL, a=altitude QN, T=QPZ, t=SPZ, and d=SP—PZ, we have [244, 247.]

Cos D . cos L : R^2 : : cos d—sin A : ver t, and in QPZ, Cos D . cos L : R^2 : : cos d—sin a : ver T.

Therefore cos D . cos L : R^2 : : sin A—sin a : ver T—ver t, whereof the first ratio is invariable, though the terms of the second vary.

PROBLEM VII.

Given three altitudes of the sun on the same day, and the intervals of time, to find the time from noon of one of the observations.

Let the arcs AN, BN, CN (see Plate II. fig. 10.) be the measures of the three times from noon, at which there are given altitudes, denoted by a, b, c (in the order in which they are observed,) and let the intervals of time, m, n, measured by AB, BC, be also given, to find the time CN of the third altitude c from noon.

Upon the diameter MN let fall the perpendiculars

AP, BQ, CR, and let BQ meet the circumference again in b, and intersect the chord AC in O. Then, by Lemma 2,

Sin a ~ sin c : sin b ~ sin c : : PR : QR : : $\frac{1}{2}$AC : $\frac{1}{2}$CO.

But the sines of the altitudes and the chord AC of a given arc are given; therefore CO is given. Hence, in the triangle BCO, we may find the angle O by these analogies,

$$\begin{cases} \frac{1}{2}\text{CO} : \frac{1}{2}\text{CB} : : \text{R} : \tan x. \quad [477.] \\ \text{R} : \tan(45+x) :: \tan\frac{1}{2}\text{BCO} : \tan(\text{O}+\frac{1}{2}\text{BCO}) :: \tan\frac{1}{2}m : \tan(\text{O}+\frac{1}{2} \end{cases}$$

Now, when the altitudes are all on the same side of noon, the angle

$$\text{O} = \tfrac{1}{2}(A b + \text{BC}) = \tfrac{1}{2}(\text{AN} + \text{NC}) = \text{AN} + \tfrac{1}{2}\text{AC}.$$

And, in other cases,

$$\text{O} = \tfrac{1}{2}(A b \sim \text{BC}) = \tfrac{1}{2}(\text{AN} \sim \text{NC}) = \text{AN} \sim \tfrac{1}{2}\text{AC};$$

therefore, universally, O + $\frac{1}{2}$AC = CN, the time of the third altitude from noon.

Otherwise,

In the triangle ABC, BO is a line drawn from the vertex B to meet the base AC (or the same produced,) therefore, [472.] cot O. AC = cot A. OC — cot C. AO. But, AC, OC, AO are proportional to PR, QR, PQ, therefore, by Lemma 2, (sin a ~ sin c) cot O = (sin b ~ sin c) . cot $\frac{1}{2}n$ — (sin a ~ sin b) . cot $\frac{1}{2}m$.

When b is the greatest altitude (or B nearest to noon,) the equation is, $(\sin a \smile \sin c).\cot O = (\sin b - \sin c)\cot \frac{1}{2}n + (\sin b - \sin a)\cot \frac{1}{2}m$, from which O being found, then $(O + \frac{1}{2}AC) = O + \frac{1}{2}(m + n)$ is the time of the third altitude from noon, as before.

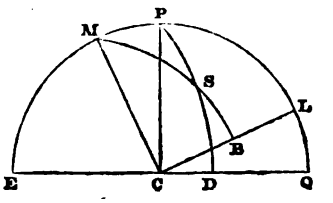
WHEN THE INTERVALS ARE EQUAL.

In this case, $m = n$, and cot A = cot C; hence, from the general equation above, we deduce this,
$(\sin a \smile \sin c) \cot O = 2\sin b \smile (\sin a + \sin c).\cot \frac{1}{2}m$,
or, $2\sin b \smile (\sin a + \sin c) : \sin a \smile \sin c :: \tan \frac{1}{2}m : \tan O$.
And here it is evident that O expresses the time of the second altitude from noon.

PROBLEM VIII.

To find the latitude and longitude of a star, from its declination and right ascension, and the obliquity of the ecliptic.

Let EQ be the equator, CL the ecliptic intersecting it in C, the first point of Aries, P, M their poles, S the star, PSD its circle of right ascension, and MSB its circle of longitude. The



declination SD, the right ascension CD or CPD, and the obliquity of the ecliptic MP or LCQ, are given to find the latitude SB, and longitude CB or CMB. Put a for the given right ascension, p for the given polar distance PS, and o for the obliquity of the ecliptic.

1. Let the right ascension extend to a point D in the first quadrant of the equator. There are given two sides, MP, PS of the triangle MPS, and the included angle, to find the angle M opposite to PS (the complement of CMB,) and the remaining side MS (the complement of SB). For which we have these analogies, [521.]

$$\text{Sin}\,\frac{p+o}{2} : \sin\frac{p\smile o}{2} :: \cot\frac{90+a}{2} : \tan x.$$

$$\text{Cos}\,\frac{p+o}{2} : \cos\frac{p\smile o}{2} :: \cot\frac{90+a}{2} : \tan y.$$

$$\text{Sin}\,x : \sin\frac{p\smile o}{2} :: \cos\frac{90+a}{2} : \sin \tfrac{1}{2}\text{MS}.$$

Then, $\left\{\begin{array}{l}\text{if } p > m,\ 90-(x+y)=\text{CB}\\ \text{if } p < m,\ 90-(x\smile y)=\text{CB}\end{array}\right\}$ the longitude required,

and $90 \smile \text{MS} = \text{SB}$ the latitude.

2. Let the right ascension extend to any point beyond the first quadrant. It is easy to find that the following changes will take place.

If it extend to a point in the second, third or fourth

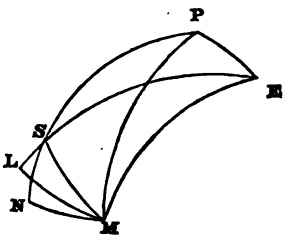
quadrant, we must substitute $\frac{270-a}{2}$ in place of $\frac{90+a}{2}$ in the analogies.

If it extend to the fourth quadrant, the equations for CB remain the same, only if CB should be negative, 360° must be added to it, or (which is the same) 450 put in place of 90. But, if it extend to the second or third, the sign — must be changed to + in the said equations.

PROBLEM IX.

The differences of right ascension and declination between a known and an unknown star being given, to find the latitude and longitude of the latter.

Let P be the pole of the equator, E the pole of the ecliptic, and S the given star, of which A, D are the right ascension and declination, and λ, L the longitude and latitude. Let M be another star, whose difference of right ascension SPM ($= a$), and difference of declination SP ~ PM ($= d$) from the former, are given; it is required to find the longitude and latitude of the lat-



ter, or to find their difference of longitude SEM, and difference of latitude SE ~ EM.

Whether the given differences a, d be great or small, the most direct, and perhaps the best method of solution is to be obtained (as in the preceding Problem) from the triangle MPE. For, if $A \angle 90$, $SPE = (90 + A)$, if $A > 90$, $SPE = 270 \sim A$; hence the angle $MPE = (A \pm a)$ is given, and the including sides EP, PM are also given, the former being the measure of the obliquity of the ecliptic, and the latter the complement of $(D \pm d)$; the base ME, therefore, (which is the complement of the required latitude,) and the angle PEM may be found. Then, from PEM $(= x \pm y)$ the longitude may be determined, as in the preceding Problem.

When the given differences are each less than a degree, the Problem may be resolved thus:

From M upon ES, SP (produced if necessary,) let fall the perpendiculars ML, MN. In the triangle MPN, the hypotenuse PM, and the angle MPN being given, we may find the side MN, and the small difference between MP, PN, or, which is the same, between SN and d; hence SN will be known. From the sides MN, NS of MNS, considered as a rectilineal triangle,

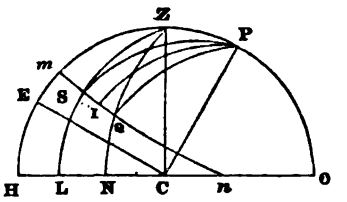
we have the angle MSN; and from the sides of the triangle SPE, which are all given, we have the angle LSN, the sum or difference of which angles gives MSL. Then, to find SL, as sin MSN : cos MSL : : MN : SL. Now, from the triangle EMS we may find the angle MEL, [325.] which is the difference of longitude; and from EML, the small difference of EL, EM, which, added to SL, if EL $\angle$ 90° (otherwise subtracted from SL,) gives the difference of latitude.

Thus, if PS $= p$, PM $-$ PS $= d$, and MPN $= a$. Sin x(PM $\sim$ PN) $= \sin^2 \frac{1}{2}a . \cos 2(p+d)$. SN $= d \pm x$. MN $= a . \sin(p+d) . \frac{\text{MN}}{\text{SN}} = \tan \text{MSN}$. The angle LSN to be found from the triangle SPE. MSL $=$ MSN $\pm$ LSN. Sin MSN : cos MSL : : MN : SL. Sin EL : sin SL : : tan MSL : tan MEL. Sin $z = \sin^2 \frac{1}{2}$MEL . sin 2EL, and SL $\pm z =$ difference of latitude.

PROBLEM X.

Given two altitudes of the sun, the intermediate time, and the declination (supposed to continue the same,) to find the latitude of the place.

Let HZO the meridian, passing through Z the zenith of the place, and through P the elevated pole, intersect the plane of the equator in the semidiameter EC, and that



of the horizon in the diameter HCO; let S, Q be two positions of the sun or a star upon the parallel of declination *mn*, and PS, PQ two hour circles, whose inclination SPQ is the measure of the time between the observations: And let ZSL, ZQN be the two vertical quadrants upon which the star is observed. There are given the altitudes SL, QN, the declination E*m*, and the hour angle SPQ, to find the latitude EZ.

Let a great circle pass through S, Q, upon which let fall the perpendicular PI. In the right-angled triangle PIQ there are given PQ, the star's polar distance, and the angle IPQ ($= \frac{1}{2}$SPQ), whence the angle PQI and the side IQ may be found, [487, 490.] Then, in the triangle ZSQ, we shall have all the sides, viz. SQ ($= 2$IQ), and ZS, ZQ the zenith distances, or complements of the altitudes, from which the angle ZQS may be determined. The difference between the angles PQI and ZQS, when the point *m* falls between Z and H, or their sum, when *m* is between Z and P,

gives the angle PQZ. And since the including sides PQ, QZ are also given in the triangle ZPQ, we may thence find the base PZ, the complement of which is the latitude.

Hence, if A, a denote the two altitudes, s half their sum, d half their difference, D the declination, t half the intermediate time, L the latitude, and $x = \mathrm{IQ}$, the radius being 1,

METHOD I.

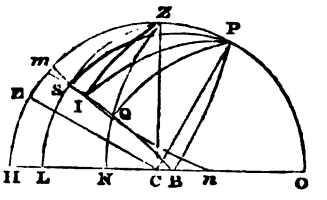
$$\text{Sin } x = \cos \mathrm{D} \,.\, \sin t.$$
$$\text{Cot PQI} = \sin \mathrm{D} \,.\, \tan t.$$
$$\text{Cos ZQS} = \sin \mathrm{A} \,.\, \sec a \,.\, \operatorname{cosec} 2x - \tan a \,.\, \cot 2x.$$
$$\text{Sin L} = \cos \mathrm{PQZ} \,.\, \cos a \,.\, \cos \mathrm{D} + \sin a \,.\, \sin \mathrm{D}.$$

The varieties of this method consist in the different ways of finding an angle from the three sides of a triangle, and a side from the other two sides and their included angle. It may be changed, for example, as follows:

$$\left.\begin{aligned} \text{Cot PQI} &= \sin \mathrm{D} \,.\, \tan t. \\ \text{Tan } z &= \cos \mathrm{PQI} \,.\, \tan s \,.\, \cot \mathrm{D}. \\ \text{Tan } y &= \cot z \,.\, \tan s \,.\, \tan d. \\ \text{Tan}^2 \tfrac{1}{2}\mathrm{SQZ} &= \cot (45+z) \,.\, \cot (45+y). \end{aligned}\right\} [296.]$$

$$\text{Tan } m = \operatorname{cosec} \tfrac{1}{2}(\mathrm{PQ}-\mathrm{QZ}) \,.\, \sin \tfrac{1}{2}\mathrm{PQZ} \,.\, \sqrt{\cos a \,.\, \cos \mathrm{D}}.$$
$$\text{Sin } \tfrac{1}{2}\mathrm{PZ} = \sin \tfrac{1}{2}(\mathrm{PQ} - \mathrm{QZ}) \,.\, \sec m. \quad [297, 299.]$$

Again, let the great circle SQ intersect the horizon at B, through which draw the hour circle PB. Then, SBL, QBN are two right-angled triangles, having a common angle at B, of which the perpendiculars SL, QN are given; also, SQ, the difference of their hypotenuses (being found as above,) is known; whence we may find IB and SBL. [341. 43.] Next, from the triangle SPB, the segments of the base SI, IB, and the side SP being given, the other side PB, and the angle PBS may be found. [327, 300.] Lastly, in the right-angled triangle POB, we shall have the hypotenuse PB and the angle PBO, for PBO is the sum of the complements of the angles SBL and PBS, when m is between Z, H; and the difference of the same, when m is between Z, P, to find the latitude PO. Hence,



METHOD II.

$$\text{Sin } x = \cos d \,.\, \sin t.$$
$$\text{Tan IB} = \tan x \,.\, \tan s \,.\, \cot d.$$
$$\text{Sin SBL} = \sin \text{A} \,.\, \text{cosec (IB} + x).$$
$$\text{Tan PBS} = \text{cosec IB} \,.\, \cot \text{D} \,.\, \cos t.$$
$$\text{Sin L} = \text{cosec PBS} \,.\, \sin \text{PBO} \,.\, \tan x \,.\, \cot t.$$

There are also varieties of this method, arising from the different ways of finding the angles SBL and PBS. The following may serve as a specimen:

Tan IPB $=$ tan s . cot d . tan t.
Tan PB $=$ sec IPB . cot D . cos t.
Tan PBS $=$ sec PB . cot IPB.
Sin SBL $=$ sin PBS . cosec (IPB $+ t$) . sin A . sec D.
Sin L $=$ sin PB . sin (SBL $\pm$ PBS).

Further, let the arc ZI be drawn; and since I is the middle of the base SQ of the triangle SZQ, whose sides are all given, we may find the said arc ZI and the angle ZIS, [352, 354.] the complement of which is the angle ZIP. But IP is known from the right-angled traingle PIQ. In the triangle PIZ, therefore, two sides ZI, IP, and the included angle ZIP will be given, to find the base PZ. Thus,

METHOD III.

Sin x $=$ cos D . sin t.
Cos IZ $=$ sin s . cos d . sec x.
Tan IP $=$ cot D . cos t.
Sin ZIP $=$ cos s . sin d . cosec x . cosec IZ.
Sin L $=$ cos ZIP . sin ZI . sin IP $\pm$ cos ZI . cos IP.

This method may likewise be varied several ways with advantage. Thus, the third equation may be omitted by substituting in the last, for sin IP, its value cot t . tan x, and for cos IP, sin D . sec x. The fourth equation also may be thrown out by substituting in the last, for cos ZIP . sin ZI, its equal sin ZQ . sin ZQS, or cos m . cos r, [298.] provided the equations for m and r be added, [297.]

$$\text{Sin } x = \cos \text{D} \cdot \sin t.$$
$$\text{Sin } m = \sin d \cdot \operatorname{cosec} x.$$
$$\text{Sin } r = \sin s \cdot \sec x.$$
$$\text{Sin L} = \begin{cases} \sin \text{D} \cdot \cos d \cdot \sin r \cdot \sec x. \\ \pm \cot t \cdot \tan x \cdot \cos m \cdot \cos r. \end{cases}$$

Or,

$$\text{Sin } x = \cos \text{D} \cdot \sin t.$$
$$\text{Sin } p = \sin \text{D} \cdot \sec x. \qquad p = \text{complement of PI.}$$
$$\text{Sin } m = \sin d \cdot \operatorname{cosec} x.$$
$$\text{Sin } r = \sin s \cdot \sec x.$$
$$\text{N} = \cos d \cdot \sin p \cdot \sin r. \qquad \text{Sin L} = \text{M} \pm \text{N}.$$
$$\text{M} = \cos p \cdot \cos m \cdot \cos r.$$

Or,
$$\text{Sin } x = \cos \text{D} \cdot \sin t.$$
$$\text{Sin } q = \sec d \cdot \cos x.$$
$$\text{Sin } r = \sin s \cdot \sec x.$$
$$\text{N} = \sin \text{D} \cdot \sin r \cdot \operatorname{cosec} q.$$
$$\text{M} = \cot t \cdot \cos r \cdot \cot q.$$
$$\text{Sin L} = \text{M} \pm \text{N}.$$

Or,

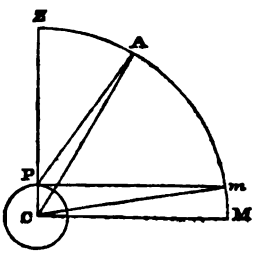
$$\text{Sin } x = \cos \text{D} \cdot \sin t.$$
$$\text{Cos}^2 x \cdot \text{N} = \sin \text{D} \cdot \sin s \cdot \cos d.$$
$$\text{Cos}^2 x \cdot \text{M} = \cot t \sqrt{\cos(s+x) \cdot \cos(s-x) \cdot \sin(x+d) \cdot \sin(x-d)}.$$

With respect to the cases in which the two quantities denoted by M, N are to be added or subtracted; if the latitude and declination be on different sides of the equator, sin L = M — N; if they be on the same side, and M greater than N, sin L = M + N; but if M be less than N, (M + N) may be the sine of one latitude, and (N — M) the sine of another, each of which will answer the conditions of the problem.

PROBLEM XI.

Given the moon's apparent altitude, with her horizontal diameter and parallax, to find her diameter and parallax at the given altitude.

Let C, M be the centres of the earth and moon. With the radius CM let a circle be described about C, draw CPZ perpendicular to CM meeting the surface of the earth in P, and the circumference of the circle in Z: Draw P*m* parallel to CM meeting the circumference in *m*, join C*m*, and to any point A in *m*Z draw CA, PA:



P*m* is the visible horizon of the place P, and CM its true horizon.

When the moon is at *m*, she has no apparent altitude to an observer at P, but her true altitude is then the angle MC*m* (or C*m*P,) which is termed her horizontal parallax. When the moon is at A, her apparent altitude is the angle AP*m*, and her true altitude the angle ACM; the difference between these, viz. the angle CAP, is her parallax in altitude.

Again, when the moon is at *m*, her apparent diameter, viewed at the distance MC (being very nearly the same with that viewed from P,) is termed her horizontal diameter: But at any altitude AP*m*, her apparent diameter, viewed from P, will exceed her horizontal diameter in the ratio of CA to AP; the excess is termed the augmentation of the moon's diameter.

Now, let a denote the moon's apparent altitude, h the horizontal parallax, p the parallax in altitude, d the horizontal diameter, and x its augmentation at the given altitude, there are given a, h and d to find p and x. From the triangles C*m*P and CAP we have,

$$\left.\begin{array}{l} 1 : \cos a :: \sin h : \sin p :: h : p. \\ \cos(a+p) : \cos a :: d : (d+x). \end{array}\right\} \begin{array}{l} \text{hence, } p = \cos a \,.\, h. \\ d+x = \dfrac{\cos a \,.\, d}{\cos(a+p)}. \end{array}$$

To determine the augmentation independently of the parallax in altitude, we have, from the last analogy,

$$\frac{x}{d+x}=1-\frac{\cos(a+p)}{\cos a}=1-\frac{\cos a-\sin a.\sin p}{\cos a}=\sin a.\sin h.$$

Therefore, $x = d$. sin a . sin h + x . sin a . sin h.

The first part of the expression being substituted for x in the second, will give a near enough approximation to the true value of x.

To find the parallax p when t the true altitude (instead of a the apparent,) is given.

Because $\sin p = \sin h$. $\cos (t - p)$.

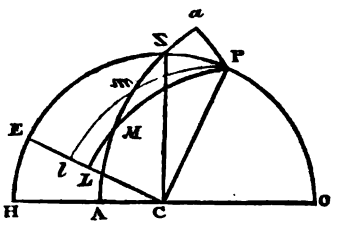
Sin $p = \sin h$. $\cos t$. $\cos p + \sin h$. $\sin t$. $\sin p$,

or, $p = \sin h$. $\cos t + p$. $\sin h$. $\sin t$.

Here, as before, we may substitute the first part of the expression for p in the second, and the whole will then be an approximation to the true value of p sufficiently near.

PROBLEM XII.

Given the latitude and longitude of the moon and zenith, with the moon's horizontal parallax, to find her parallaxes in longitude and latitude, in right ascension and declination, and the augmentation of her semidiameter.



Let HO be the horizon, Z the zenith, EC the ecliptic, P its pole, *m* the true place of the moon, M the apparent place in the same vertical circle Z*m*A, and PZE, P*ml*, PML circles of longitude through Z, *m*, M.

Put the latitude of the zenith EZ $= \lambda$, the moon's true latitude *lm* $= l$, her apparent latitude LM $=$ L, and their difference of longitude EPL or E*l* $=$ D: Also, let the parallax in longitude LP*l* or L*l* $= y$, and the parallax in latitude (PM — P*m*) $= x$.

---

1. TO FIND THE PARALLAX IN LONGITUDE.

Because sin MP*m* : sin M*m* : : sin M : sin P*m*.
And sin MZ : sin MPZ : : sin PZ : sin M.
Also, R : sin MZ : : sin h : sin M*m*.

Therefore, sin MP*m* : sin MPZ : : sin h . sin PZ : sin P*m*.

That is, sin y : sin (D + y) : : sin h . cos λ : cos l,
or, sin y . cos l = sin h . cos λ . sin (D + y), from which equation the value of y may be had by approximation.

Or thus:

Since $\frac{\cos l}{\sin h.\cos\lambda} = \frac{\sin(D+y)}{\sin y} = \sin D.\cot y + \cos D$, [29.]

therefore, $\cot y = \frac{\cos l}{\sin D.\sin h.\cos\lambda} - \cot D = \cot z - \cot D$

(suppose.)

Then, $\tan z = \sin D \,.\, \sin h \,.\, \cos\lambda \,.\, \sec l$, and,

$\tan y = \sin D \,.\, \sin z \,.\, \operatorname{cosec}(D - z)$.

Or thus:

Putting $\frac{\sin h.\cos\lambda}{\cos l} = Q$; then $\sin y = Q.\sin(D+y)$

$= Q.\sin D.\cos y + Q.\cos D.\sin y$; hence, putting $Q.\cos D = n$, $\tan y = n.\tan D + n.\tan y$. Thus, we have,

$$n = \sin h.\cos\lambda.\sec l.\cos D,$$

$$\text{and } \tan y = \frac{n}{1-n}.\tan D.$$

Or thus:

Since $\sin(D+y) : \sin y :: \cos l : \sin h.\cos\lambda :: R : \tan z$ (suppose,) therefore $R : \tan(45+z) :: \tan\frac{1}{2}D : \tan(y+\frac{1}{2}D)$,

to which adding $\frac{1}{2}$D, the sum is the apparent difference of longitude, or, subtracting it, the remainder is the parallax in longitude.

---

2. TO FIND THE APPARENT LATITUDE.

We have [348.] $\tan L \cdot \sin D + \tan \lambda \cdot \sin y = \tan l \cdot \sin(D+y)$. Therefore $\tan L = \operatorname{cosec} D \cdot \tan l \cdot \sin(D+y) - \operatorname{cosec} D \cdot \tan \lambda \cdot \sin y$, which equation gives the natural tangent of the apparent latitude. But if the moon's latitude be under 1°, it will give the apparent latitude itself, as in that case the small arcs L, l, y may be taken in place of their tangents or sines.

The last term becomes affirmative when the moon and zenith are on different sides of the ecliptic. If the moon and zenith be on the same side of the ecliptic, and the last term the greater, the moon's true and apparent places are on different sides of the ecliptic, in other cases they are on the same side.

Or thus:

$$\frac{\text{Tan L} \cdot \sin D}{\sin y} = \frac{\tan l \cdot \sin (D+y)}{\sin y} - \tan \lambda.$$

Put $\tan z = (\tan l \cdot \sin(D+y) \cdot \operatorname{cosec} y =) \sin l \cdot \sec \lambda \cdot \operatorname{cosec} h$.

Then $\tan L = \sin (z \mp \lambda) \cdot \sin y \cdot \sec z \cdot \sec \lambda \cdot \operatorname{cosec} D$.

It is $(z + \lambda)$ when the moon and zenith are on different sides of the ecliptic, and $(z - \lambda)$ when they are on the same side: And, in the latter case, when $\lambda > z$ the true and apparent places of the moon are on different sides.

Or thus:

Let fall the perpendicular PQ upon mZ (produced if necessary;) then, putting P = PZ, and p = Pm,

Sin P + p : sin P − p : : cot $\frac{1}{2}$D : tan Z. [332.]

And cos (Z ± $\frac{1}{2}$D) : cos (Z ± $\frac{1}{2}$(D + y)) : : tan l : tan L. [327.]

Or thus:

Let t, a be the moon's true and apparent altitudes, and d their difference (or parallax in altitude,) then,

Sin $\frac{1}{2}$(P + p) : sin $\frac{1}{2}$(P − p) : : cot $\frac{1}{2}$D : tan z. [284.]

Sin z : sin $\frac{1}{2}$(P − p) : : cos $\frac{1}{2}$D : sin $\frac{1}{2}m$Z. [292.]

And $\frac{1}{2}m$Z subtracted from its complement gives t.

Now, d = sin h . cos t + d . sin h . sin t, and a = t − d.

Hence, sin L = sec t . cos a . sin l − sec t . sin d . sin λ, [347.] which equation gives the natural sine of the apparent latitude, or the apparent latitude itself, if it do not exceed a degree.

3. TO FIND THE PARALLAX IN LATITUDE.

Because $\tan l . \sin(D+y) = \tan\lambda . \sin y + \tan L . \sin D$,
$(\tan l - \tan L).\sin(D+y) = \tan\lambda.\sin y - \tan L(\sin(D+y) - \sin D)$,
hence,

$$\frac{\sin x}{\cos l.\cos L} = \tan\lambda \frac{\sin y}{\sin(D+y)} - \tan L \frac{2\sin\frac{1}{2}y.\cos(D+\frac{1}{2}y)}{\sin(D+y)}.$$

Instead of $\frac{\sin y}{\sin(D+y)}$ or $\frac{2\sin\frac{1}{2}y}{\sin(D+y)}$, let its value found before, viz. $\frac{\sin h \cos\lambda}{\cos l}$, be substituted, and let all the terms be multiplied by $\cos l . \cos L$; then,

$\mathrm{Sin}\, x = \sin h . \sin\lambda . \cos L - \sin h . \cos\lambda . \cos(D+\frac{1}{2}y) . \sin L$,

whence x the parallax in latitude may be found by approximation.

The last term becomes affirmative when the moon's polar distance and $(D + \frac{1}{2}y)$ are unlike.

Or thus:

From the triangle ZPm find the angle m as well as the base mZ, and thence the parallax in altitude Mm $(= d)$: then letting fall the perpendicular Mr upon Pm, and putting $Mr = v$, $mr = z$, we have,

R : sin m : : sin d : sin v.

R : cos m : : tan d : tan z.

(and cos $l-z$: R : : tan z : tan y, parallax in longitude.)

Also, because $\tan \frac{d+z}{2} . \tan \frac{d-d}{2} = \tan \frac{MP+Pr}{2}$ $. \tan \frac{MP-Pr}{2}$, $(MP-Pr=)\delta=(d-z).\tan\frac{d+z}{2} .\tan(l-z)$.

Hence the parallax in latitude is $x=y\pm\delta$; + or — according as the moon and zenith are on the same or on different sides of the ecliptic.

If the moon's true place be in the ecliptic,

Cos T : sin λ : : sin d : sin x, the apparent latitude being on the other side of the ecliptic in respect to the zenith.

And tan T . tan D . tan d = tan y, the apparent increase of the moon's longitude from the zenith.

In the same way may be found the sun's apparent latitude (allowing for both parallax and refraction, of which the latter is the greater,) on the same side of the ecliptic with the zenith, and his apparent decrease of longitude from the zenith.

4. TO FIND THE PARALLAXES IN RIGHT ASCENSION AND DECLINATION.

Let EC be the equator, and P its pole, Z the zenith, and HO the horizon, &c. then EZ $= \lambda$ will be the declination of the zenith (or latitude of the place); l, L the true and apparent declinations of the moon; D, $(D+y)$ the true and apparent differences of right ascension between the moon and zenith. So that *(mutatis mutandis,)* the same theorems may be applied to find these parallaxes.

5. TO FIND THE AUGMENTATION OF THE MOON'S SEMIDIAMETER.

Put the semidiameter $= s$, and its augmentation $= z$; then, $(s+z) : s :: \sin MZ : \sin mZ :: \sin MPz \cdot \sin MP : \sin mPZ \cdot \sin mP$.

Therefore,

$$\frac{s+z}{s} = \frac{\sin(D+y).\cos L}{\sin D \cdot \cos l} = \frac{\sin(D+y)}{\sin D} \cdot \frac{\cos l \mp x}{\cos l}.$$

But $\frac{\sin(D+y)}{\sin D} = \cos y + \cot D \cdot \sin y = 1 + m.$

In like manner, $\frac{\cos(l \mp x)}{\cos l} = 1 \pm \sin x . \tan l = 1 \pm n.$

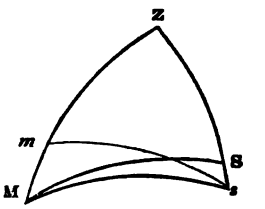
Hence, $z = s \cdot (m \pm n \pm mn)$.

The last two terms being negative when the moon and zenith are on different sides of the ecliptic.

In the computation of solar eclipses, the true and apparent latitudes of the moon are so small, that their cosines may be supposed equal; hence, $\frac{s+z}{s} = \frac{\sin(D+y)}{\sin D}$, which gives an easy rule for finding the moon's augmented semidiameter, independently of her altitude.

PROBLEM XIII.

Given the apparent distance, and the apparent altitudes of the centres of the sun and moon, with the moon's horizontal parallax, to find their true distance.



Let SZ, MZ be the apparent zenith distances, or the complements of the apparent altitudes A, a of the centres of the sun and moon, and SM the apparent distance of the same; it is required to find the true distance.

The sun being more elevated by refraction than depressed by parallax, the true place of its centre will be at a point s, in the same vertical circle ZS, below the point S. But the moon being more depressed by parallax than elevated by refraction, the true place of its centre will be at m, in the same vertical circle ZM, above the apparent place M.

Now Ss is found by subtracting the sun's parallax in altitude from the refraction corresponding to the altitude A; or, in the case of a star, Ss is simply the refraction: And Mm is found by subtracting the refraction at the altitude a from the moon's parallax in altitude. Thus, Mm, mZ, ZS, Ss and SM are all given to find ms.

METHOD I.

The most obvious method of solution is by a simple application of Theorem 347 to the triangle ZMs, formed by drawing an arc through the points M, s. The true altitudes of the sun and moon being denoted by T, t; put SM = D, Ss = s, and Mm = m; then,

$$1.\quad \begin{cases} \text{Cos M}s \text{ . cos A} = \text{cos D . cos T} - \text{sin } a \text{ . sin } s. \\ \text{Cos } ms \text{ . cos } a = \text{cos M}s \text{ . cos } t + \text{sin T . sin } m; \end{cases}$$

where, if D be greater than 90°, cos D . cos T in the first equation, and consequently cos Ms . cos t in the second, will be negative.

Or, if the arc be drawn through m, s, and the same theorem be applied, first to the triangle ZSM, and next to the triangle Zms, we shall have,

2. $$\begin{cases} \text{Cos } m\text{S} \cdot \cos a = \cos \text{D} \cdot \cos t + \sin \text{A} \cdot \sin m. \\ \text{Cos } ms \cdot \cos \text{A} = \cos m\text{S} \cdot \cos \text{T} - \sin t \cdot \sin s. \end{cases}$$

And if, in each of these forms, the value of the unknown quantity in the former equation be substituted for the said quantity in the latter, we shall have two other forms, viz.

3. $$\text{Cos } ms = \frac{\cos \text{D} . \cos \text{T} . \cos t}{\cos \text{A} . \cos a} - \frac{\sin a . \cos t . \sin s}{\cos \text{A} . \cos a} + \frac{\sin \text{T} . \sin m}{\cos a}.$$

4. $$\text{Cos } ms = \frac{\cos \text{D} . \cos \text{T} . \cos t}{\cos \text{A} . \cos a} + \frac{\sin \text{A} . \cos \text{T} . \sin m}{\cos \text{A} . \cos a} - \frac{\sin t . \sin s}{\cos \text{A}}.$$

Half the sum of these gives the fifth form.

5. $$\text{Cos } ms = \frac{\cos \text{D} . \cos \text{T} . \cos t + \frac{1}{2}\sin(2\text{A} - s) . \sin m - \frac{1}{2}\sin(2a + m) . \sin s}{\cos \text{A} . \cos a}.$$

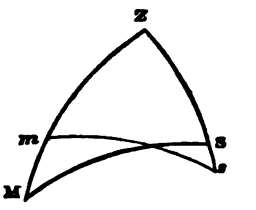
In these forms the cosine only of the true distance being found, this method will be inaccurate when the distance is small; but in that case we may change the first form, for instance thus:

Put x for the small difference of MS, ms and Ms' for half their sum (being nearly = Ms,) then subtracting both sides of the second equation from cos Ms . cos a, we have cos a . sin Ms' . sin x = cos Ms . sin $(a + \frac{1}{2}m)$. sin m — sin T . sin m, from which x, and consequently ms may be accurately found from Ms; but Ms must

be first found, in like manner, from the first equation: thus, if z be the difference of D, Ms, and D' half their sum, then $\cos A . \sin D . z = \cos D . \sin (A - \frac{1}{2}s) . s - \sin a . s$.

METHOD II.

Another obvious method of proceeding in the solution of this problem is, to find first the vertical angle Z of the triangle MZS, of which all the sides are given, and then the base ms of the triangle mZs, of which the other two sides containing the angle found are given.



From the different ways of resolving these cases, which we have shewn before, various forms of this method may be deduced. Thus, putting x for the complement of half the difference of the angles m, s at the base of the triangle mZs,

$$\text{Sin } \tfrac{1}{2}(T - t) : \cos \tfrac{1}{2}(T + t) :: \tan \tfrac{1}{2}Z : \tan x.$$

$$\text{Sin } x : \cos \tfrac{1}{2}(T + t) :: \sin \tfrac{1}{2}Z : \sin \tfrac{1}{2}ms.$$

Now both $\tan \frac{1}{2}Z$ and $\sin \frac{1}{2}Z$ may be obtained from the triangle MZS as follows (putting $s = \frac{1}{2}(D + A + a)$);

$$\sqrt{\cos S . \cos (S - D)} : \sqrt{\sin (S - A) . \sin (S - a)} :: R : \tan \tfrac{1}{2}Z.$$

$$\sqrt{\cos A . \cos a} : \sqrt{\sin (S - A) . \sin (S - a)} :: R : \sin \tfrac{1}{2}Z.$$

Hence, putting N, P, Q for the three radical quantities, we have the following forms:

1. $\begin{cases} \text{Sin } \frac{1}{2}(T \sim t).N : \cos \frac{1}{2}(T+t).P :: \text{rad} : \tan x. \\ \text{Sin } x \,.\, Q \quad : \cos \frac{1}{2}(T+t).P :: \text{rad} : \sin \frac{1}{2}ms\,; \end{cases}$

or,

2. $\text{Cos } x \,.\, Q \quad : \sin \frac{1}{2}(T \sim t).N :: \text{rad} : \sin \frac{1}{2}ms\,;$

3. $\begin{cases} \text{Sin } \frac{1}{2}(T+t).P : \cos \frac{1}{2}(T \sim t).N :: \text{rad} : \tan y. \\ \text{Sin } y \,.\, Q \quad : \cos \frac{1}{2}(T \sim t).N :: \text{rad} : \cos \frac{1}{2}ms\,; \end{cases}$

or,

4. $\text{Cos } y \,.\, Q \quad : \sin \frac{1}{2}(T+t).P :: \text{rad} : \cos \frac{1}{2}ms.$

Again, since

Ver (MZ + ZS) — ver (MZ ~ ZS) : 2R : : ver MS — ver (MZ ~ ZS) : ver Z, and 2R : ver (mZ + Zs) — ver (mZ ~ Zs) : : ver Z : ver ms — ver (mZ ~ Zs).

Therefore,

Suver (A+a) — ver (A ~ a) : suver (T+t) — ver (T ~ t)
: : ver D — ver (A ~ a) : ver ms — ver (T ~ t).
: : suver (A+a) — ver D : suver (T+t) — ver ms,

which analogies may be thus transformed:

Cos A . cos a : cos T . cos t
: : sin (S — A).sin (S — a) : $\sin^2 \frac{1}{2}ms$ — $\sin^2 \frac{1}{2}(T \sim t)$.
: : sin (S — A).sin (S — a) : $\cos^2 \frac{1}{2}(T \sim t)$ — $\cos^2 \frac{1}{2}ms$.
: : cos S . cos (S ~ D) : $\cos^2 \frac{1}{2}(T+t)$ — $\sin^2 \frac{1}{2}ms$.
: : cos S . cos (S ~ D) : $\cos^2 \frac{1}{2}ms$ — $\sin^2 \frac{1}{2}(T+t)$.

Hence, putting $\sqrt{\frac{\cos A \cdot \cos a}{\cos T \cdot \cos t}} = M$, and N, P as before, we have the following forms adapted to practice, viz.

5. $\begin{cases} \text{Tan } x = M \cdot P \text{ cosec } \frac{1}{2}(T \sim t). \\ \text{Sin } \frac{1}{2}ms = \sec x \cdot \sin \frac{1}{2}(T \sim t). \end{cases}$

6. $\begin{cases} \text{Sin } y = M \cdot P \cdot \sec \frac{1}{2}(T \sim t). \\ \text{Cos } \frac{1}{2}ms = \cos y \cdot \cos \frac{1}{2}(T \sim t). \end{cases}$

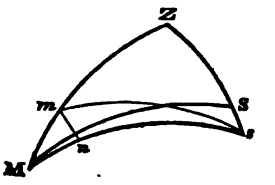
7. $\begin{cases} \text{Sin } v = M \cdot N \cdot \sec \frac{1}{2}(T + t). \\ \text{Sin } \frac{1}{2}ms = \cos v \cdot \cos \frac{1}{2}(T + t). \end{cases}$

8. $\begin{cases} \text{Tan } z = M \cdot N \cdot \text{cosec } \frac{1}{2}(T + t). \\ \text{Cos } \frac{1}{2}ms = \sec z \cdot \sin \frac{1}{2}(T + t). \end{cases}$

METHOD III.

A third method is that of approximation, which consists in finding the small difference between the true and apparent distances.

Thus, in the triangle MZS, the side MZ and the angle Z being fixed, and the side ZS varying by the very small increment S*s*, we may find the increment or decrement of MS, by this analogy,



$$R : \cos ZSM :: s : x. \quad [439.]$$

the cos ZSM being given in terms of MZ, ZS and SM

[266.] Hence MS will be known, and the other sides of the triangle MZ*s* being given, the cos ZM*s* is given. Hence, if the arc *mn* be perpendicular to M*s*, R : cos ZM*s* : : tan M*m* : tan M*n* : : *m* : *y*, from which *y* (= M*n*), and consequently *ns* is given. Now *ms* being nearly equal to *ns*, the small difference between them (*z*) may be found thus :

$$\text{Tan}\,\tfrac{1}{2}(m+y).\tan\tfrac{1}{2}(m-y) = (\tan^2\tfrac{1}{2}mn = \tan\tfrac{1}{2}(ms+sn) \cdot \tan\tfrac{1}{2}(ms-sn) =) \tan ns \cdot \tan\tfrac{1}{2}z.$$

Hence the following form :

1.
$$\begin{cases} x = \sin a.\sec \text{A}.\text{cosec D}s - \tan \text{A}.\cot \text{D}.s. & \text{D} \pm x = \text{D}'. \\ y = \sin \text{A}.\sec a.\text{cosec D}'.m - \tan a.\cot \text{D}.m. & \text{D} \pm y = \text{D}''. \\ z = \tfrac{1}{2}(m+y)\,(m-y)\cot \text{D}''. \quad \text{And} & \text{D} \pm z = ms. \end{cases}$$

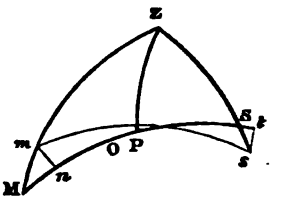
Observe, 1. D + *x* = D', or D − *x* = D', according as the first term is greater or less than the second.

2. But when D is greater than 90°, the second term becomes affirmative, and in that case D + *x* = D'.

3. The same rules are to be applied in finding D''.

4. If D'' be less than 90°, D'' + *z* = *ms* ; if greater, D − *z* = *ms*.

Again, upon the base MS of the triangle MZS let fall the perpendiculars *mn*, *st* and ZP. The segments MP, PS may be found from the given sides of the triangle MZS. Then MPZ, M*nm* being two right-angled triangles, having a common angle M, about which the sides MZ, MP and M*m* being known, the other side M*n* will also be known. In like manner may *st* be found. Consequently *nt*, which is nearly equal to *ms*, will be known. Their small difference (*z*) may be found thus: Let the arcs *mn*, *st*, when produced, meet in the point Q (forming an angle at that point of which *nt* is the measure); then, from the triangle *ms*Q, we have,



$$\text{Cos Q} \cdot \sin mQ \cdot \sin sQ - \cos mQ \cdot \cos sQ = \cos ms;$$

or, $\cos nt \cdot \cos mn - \sin mn \cdot \sin st = \cos ms$. Hence,

$$\cos nt \cdot 2\sin^2 \tfrac{1}{2}mn + \sin mn \cdot \sin st = \cos nt - \cos ms$$
$$= \sin nt \cdot \sin z.$$

$$\text{And } \sin z = 2\sin^2 \tfrac{1}{2}mn \cdot \cot nt + \sin mn \cdot \sin st \cdot \operatorname{cosec} nt.$$

Hence this second form :

2. $\left\{\begin{array}{l} \text{Tan OP} = \cot\frac{1}{2}(\text{A}+a).\tan\frac{1}{2}(\text{A}\sim a)\cot\frac{1}{2}\text{D}. \\ \qquad\qquad\qquad\

$$3.\begin{cases} \text{Sin}\tfrac{1}{2}x.\sin \text{D}' = \dfrac{\sin(\text{S}-\text{A}).\sin(\text{S}-a)}{\cos \text{A}.\cos a}.\sin(\text{A}+a+q)\sin q. \\ \qquad + \dfrac{\cos \text{S}.\cos(\text{S}\sim\text{D})}{\cos \text{A}.\cos a}.\sin(a-\text{A}+p)\sin p. \\ \text{or } (-\dfrac{\cos \text{S}.\cos(\text{S}\sim\text{D})}{\cos \text{A}.\cos a}.\sin(\text{A}-a-p)\sin p). \end{cases}$$

Again, because,

Cos A . cos a : cos T . cos t : : suvers (A+a)— vers D
: suvers (T + t) — vers ms,

putting the former ratio = 1 : (1 — v), we have,

1 : v : : suvers (A + a) — vers D :
(suvers (A+a)—suvers (T+t))—(vers D—vers ms).

And (vers D–vers ms)=(suvers (A+a)–suvers (T+t))
— v (suvers (A + a) — vers D).

Hence, putting x, D' and q as above.

---

4. $\frac{1}{2}x$. sin D' = sin (A+a+q) . q — cos S.cos (S$\sim$D) . v.

The small quantity v may be thus determined,

Suppose cos A : cos (A — s) : : 1 : 1 + y.

And cos a : cos (a + m) : : 1 : 1 — z.

Then 1 : 1—v (: : cos A.cos a : cos(A—s).cos(a+m))
= 1 + y — z (1 + y).

But 1 : y : : cos A : cos (A — s) — cos A
: : cos A : sin (A — $\frac{1}{2}s$) . s.

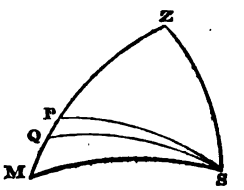
$$\text{And } 1 : z : : \cos a : \cos a - \cos (a + m)$$
$$: : \cos a : \sin (a + \tfrac{1}{2}m)\, m.$$

$$\text{Therefore } v = ((z(1+y) - y) =) \frac{m \cdot \sin (a + \frac{1}{2}m)}{\cos a} \cdot \frac{\cos A - s}{\cos A} - \frac{s \cdot \sin A - \frac{1}{2}s}{\cos A}.$$

N. B. Log. $\frac{\cos A - s}{\cos A} = 0.000120$, when A is not less than 25°; in other cases the logarithms may be taken from a small table. Also, $\frac{s \cdot \sin (A - \frac{1}{2}s)}{\cos A} = 57''$, very nearly in all cases for a star, but for the sun $9''.\sin A$ must be subtracted from $57''$.

PROBLEM XIV.

To find the correction of the computed distance of the moon and a star, arising from an error in the moon's horizontal parallax.



Let s be the true place of the star, M the apparent place of the moon's centre, MP ($= p$) being the correction of altitude answering to the horizontal parallax h, and MQ ($= q$) the correction of altitude correspond-

ing to the true horizontal parallax h; there are given MP, MQ, Ms and Ps to find Qs.

$$\text{Cos Q}s \,.\, \sin \text{MP} = \cos \text{P}s \,.\, \sin \text{MQ} + \cos \text{M}s \,.\, \sin \text{PQ}.$$

[347.]

or, $\text{Cos Q}s \,.\, p = \cos \text{P}s \,.\, q + \cos \text{M}s \,.\, (p - q)$.

From both sides, subtracting cos Ms . p, we have,

$$(\text{Cos Q}s - \cos \text{M}s)\, p = (\cos \text{P}s - \cos \text{M}s)\, q,$$

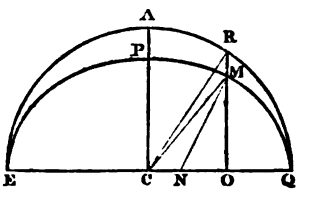
$$\text{or, } \sin \tfrac{1}{2}(\text{M}s + \text{Q}s)\,(\text{M}s - \text{Q}s)\, p = \sin \tfrac{1}{2}(\text{M}s + \text{P}s)\,(\text{M}s - \text{P}s)\, q.$$

Hence, since the leading quantities $\frac{1}{2}$(Ms + Qs) and $\frac{1}{2}$(Ms + Ps) are very nearly equal, it appears that (Ms—Qs) and (Ms — Ps) the corrections of the distance are very nearly proportional to p, q the corrections of the altitude, or to the horizontal parallaxes h', h.

PROBLEM XV.

Supposing the earth a spheroid, of which the equatorial diameter is to the axis as 1 : a; required the semidiameter at a given latitude, with the reduction of that latitude, and the consequent reduction of the moon's equatorial horizontal parallax.

Let C be the earth's centre, EPMQ the elliptic meridian of any place M, passing through the pole P, and intersecting the equator in the diameter EQ; join MC, and let MN perpendicular to the earth's surface at M meet EQ in N; also, describe the semicircle EAQ upon EQ, draw RMO perpendicular to EQ and meeting EAQ in R, and join CR.



The angle MNO (= L) is the latitude obtained by observation (being the angular distance of the zenith of the place M from the equator,) and differs from the angle MCQ, which would be the latitude, were M on the surface of a sphere whose centre is C, by the angle CMN (= x) which is called the reduction of latitude.

From the properties of the ellipse, we have $CA^2 : CP^2 :: CO : NO$; hence $CA^2 : CP^2 :: \tan MNO : \tan MCO$, that is,

$$1 : a^2 :: \tan L : \tan L - x.$$

Again, from the preceding analogy, we have,

$1 : (1 - a^2)\ (c^2) :: \tan L : \tan L - \tan (L - x) :: \sin L \cdot \cos (L - x) : \sin x$; hence,

$$\sin x = c^2 \cdot \sin L \cdot \cos (L - x).$$

And, since $\tan MNO : \tan MCO :: CA^2 : CP^2 :: RO^2 : MO^2 :: \tan^2 RCO : \tan^2 MCO$, if MCR = z,

then $\tan L : \tan (L - z) :: \tan (L - z) : \tan (L - x)$. Hence,

$$1 : a :: \tan L : \tan (L - z).$$

And $\sin z = (1 - a) \cdot \sin L \cdot \cos (L - z)$.

Now, from the triangle CRM, CR : CM :: sin CMR : sin CRM :: cos MCO : cos RCO; that is,

$1 : CM :: \cos(L - x) : \cos(L - z) :: \sin x : (1 + a) \cdot \sin z$.

Also, because $1 : CM^2 :: \sec^2 (L - z) : \sec^2 (L - x)$ $:: 1 + \tan^2 (L - z) : 1 + \tan^2 (L - x) :: 1 + \tan L \cdot \tan (L - x) : 1 + \tan^2 (L - x) :: \cot (L - x) + \tan L : \cot (L - x) + \tan (L - x) :: [36.] \cos x \cdot \cos (L - x) : \cos L :: \sin x \cdot \cos x : c^2 \sin L \cdot \cos L$,

$$CM = c \sqrt{\frac{\sin 2L}{\sin 2x}}.$$

From one of the preceding analogies, we have [47.] $\sin (2L - z) : \sin z :: \sin (2L - x - z) : \sin (x - z)$, whence it is easy to see that x, which is a very small arc, must be very nearly $= 2z$, at least when the latitude is considerable. Now, if h be the equatorial horizontal parallax, and y its reduction, $h : h - y :: \cos(L - x) : \cos (L - \frac{1}{2}x) :: \cos (L - x) : \sin (L - \frac{3}{4}x) \cdot \sin \frac{1}{2}x :: 1 : \frac{1}{2}c^2 \sin L \cdot \sin (L - \frac{3}{4}x)$.

Hence $y = \frac{1}{2}hc^2 \cdot \sin^2 L$, nearly.

Lastly, to find the reduction of latitude when a maximum.

Because $CA^2 : CP^2 : :$ tan L : tan $(L - x)$.

$CA^2 + CP^2 : CA^2 - CP^2 : :$ sin $2L - x$: sin x;

where it is evident that x is a maximum, when $(2L - x) = 90^\circ$; if therefore r be the greatest reduction, $L = 45^\circ + \frac{1}{2}r$, and since

$CA^2 : CP^2 : : \tan(45 + \frac{1}{2}r) : \tan(45 - \frac{1}{2}r) : : 1 : \tan^2(45 - \frac{1}{2}r)$,

if AQ, PQ be drawn, CPQ will be equal to the latitude, whose reduction is a maximum, the angle PQA being equal to half that reduction, and CQP the reduced latitude.

PROBLEM XVI.

Given the right ascensions and declinations of the moon and a fixed star, to find their apparent distance, as seen from a given place on the earth, considered as an oblate spheroid.

Let G be the given place on the spheroidal earth EPQ, (see Plate II. fig. 11.) of which C is the centre, EQ the equator, and P the pole. Let Gc perpendicular to the earth's surface, and consequently to the horizon $HcCO$, be continued to meet the heavens in Z, the apparent zenith of the given place. Also, let CG be produced to meet the heavens in g, which (in order to

reduce things to the same method of computation as if the earth were a sphere,) may be termed the true zenith, as the angle ECG is, on the spherical hypothesis, the true latitude of the place G. The difference between the apparent latitude ECZ (viz. the latitude obtained by observation,) and the true latitude ECG is the angle CG*c* or *g*CZ, which, in the preceding problem, was termed the reduction of latitude.

Here let it be observed and kept in mind, that two straight lines, proceeding from C, *c*, or from any two points of the earth, parallel to each other, tend to the same point (*z*) in the heavens, by reason of the immense distance of the fixed stars from the earth: Also, that any point of the moon's orbit may, for the same reason, be accounted the centre of the celestial sphere: And, that it is only the proximity of the moon to the earth that occasions any deviation from a spherical form in the latter to be taken into account, when we are to compute with accuracy the apparent position of the former.

Let L be the centre of the moon in her orbit, through which, from the several points C, *c*, G, draw the straight lines CLM, *c*L*m*, GLM to meet the heavens (or surface of the celestial sphere,) at M, *m*, M; of which M marks the true place of the moon, seen from the centre of the earth as a spheroid; *m* the true place,

seen from the centre of the earth as a sphere, and M the apparent place, as seen from the given point G, on either hypothesis. The planes M'LCGg and mLcGZ are the planes of two great circles, which intersect each other in the line GLM (or, which is the same, a line parallel to GLM through C,) and whose circumferences intersect at M, the moon's apparent place. Now to find the moon's true zenith distance M'g, and azimuth M'gp; there are given M'p, the moon's polar distance, pg the complement of the reduced declination of the zenith, and the included angle M'pg, the difference of their right ascensions. In like manner may the zenith distance Sg, and azimuth Sgp, of the fixed star S, be found. Then, in the triangle M'gS, there will be given two sides and the included angle, from which the base M'S (the true distance of the moon and star,) may be found. Thus, we have only to find M'M, in order to obtain the apparent distance M, in like manner, from the triangle MgS.

Since LC, Lc are very nearly equal, we have,

$$\text{Sin M'M} : \sin \text{M}m :: \sin \text{CLG} : \sin c\text{LG}$$
$$:: \sin \text{LGC} \cdot \text{GC} : \sin \text{LG}c \cdot \text{G}c.$$

Therefore, $\sin$M'M : $\sin$Mm :: $\sin$Mg.rad : $\sin$MZ.$\cos gZ$.

But $\sin$ Mm : $\sin$ h : : $\sin$ MZ : rad (h = horizontal parallax to GC or Gc).

Hence $\sin$ M'M : $\sin$ h : : $\sin$ Mg : $\cos$ gZ.

But gZ is, in all cases, so small an arc, that its cosine may, without sensible error, be esteemed equal to the radius: so that M'M may be found from M'g, in the same manner as though M'g were the moon's apparent zenith distance, with sufficient accuracy.

Thus it appears that we have only to reduce the latitude of the place, (or more properly the declination of the zenith,) and the moon's horizontal parallax, as shewn in the preceding problem, and then make the same computation as if the earth were a sphere.

Otherwise,

From the preceding solution, it is plain that the apparent place of the moon, as seen from G on the spheroid, having its centre at C, instead of G on the sphere having its centre at c, is affected only by a diminution of parallax, and a change of azimuth. The parallax of altitude M'M is diminished by the horizontal parallax being reduced in the ratio CV : CG. And the moon's true or apparent azimuth, on the spherical hypothesis, mZp is changed to M'Zp, in consequence of the true place at m being transferred to M'; for the two lines which proceed from the points C, c to the moon's centre at L, contain an angle of sensible magnitude.

Now, to find the variation of azimuth, viz. the small angle M'Z*m*; in the triangle M*Zg*, in which the arc ZM' is drawn from the vertex to the base, we have, [348.]

Cot M'Z.sin M*Zg* = cot MZ.sin M'*Zg* + cot *g*Z.sin M'*Zm*,

that is, supposing the moon's true and apparent altitudes to be denoted by *t*, *a*, the true azimuth M*Zg* by *m*, the reduction of declination *g*Z by *r*, and the variation of azimuth M'Z*m* by *x*,

$$\tan t \,.\, \sin m = \tan a \,.\, \sin (m - x) + \cot r \,.\, \sin x.$$

But x is so very small a quantity, that sin m and sin $(m - x)$ do not materially differ. Hence,

$$\sin x = \tan r \,.\, \sin m \,(\tan t - \tan a).$$

And, putting p for the parallax in altitude, and h for the horizontal parallax,

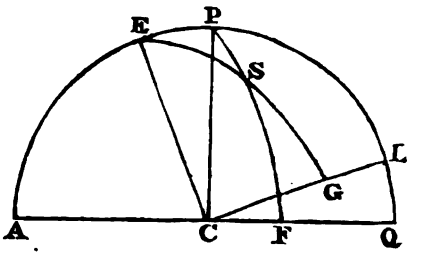
$$\sin x = \tan r \,.\, \sin m \,.\, \sin p \,.\, \sec a \,.\, \sec t, \quad [34.]$$

$$\text{or, } x = h \,.\, \tan r \,.\, \sin m \,.\, \sec t.$$

The variation of azimuth being thus found and deducted from M'Z*p*, the moon's true azimuth from the pole, we have the azimuth M*Zp*. Hence the difference of azimuth MZS, as well as the containing sides MZ, ZS of the triangle MSZ may be obtained, and thence the apparent distance MS.

PROBLEM XVII.

To find the variation in the right ascension and declination of the stars, arising from the precession of the equinox.



Let P be the pole of the equator ACQ, E the pole of the ecliptic CL, and S a star, whose circle of longitude is ESG, and circle of right ascension PSF.

The pole of the equator has a very slow motion round the pole of the ecliptic, in a direction contrary to the order of the signs; in consequence of which the equinoctial points move along the ecliptic, in the same direction, at the rate of $50''\frac{1}{4}$ annually, thereby increasing the longitude of every fixed star by the same quantity, while the latitude remains the same.

Thus, in the triangle EPS, the sides EP, ES remaining fixed, the variation of the angle E is given to find those of the angle P and the side PS. Let A, D denote the star's right ascension and declination, O the obliquity of the ecliptic, and $a = 50''\frac{1}{4}$: Then, since $E'' : P :: 1 : \cos EP - \sin EP \cdot \cos P \cdot \cot PS$. [412.]

And $E'' : PS'' :: 1 : \sin EP . \sin P$. [414.]

We have the following formulæ for the variations in right ascension and declination.

$$A'' = a . \cos O + a . \sin O . \sin A . \tan D.$$

$$D'' = a . \sin O . \cos A.$$

PROBLEM XVIII.

To find the variation in the longitude of the stars, and in the obliquity of the ecliptic, arising from the nutation of the earth's axis.

Let E (see Plate II. fig. 12.) be the pole of the ecliptic, and P the mean place of the pole of the equator, about which the true pole *t* describes a very small ellipse *abcd*, the greater semiaxis *a*P, on the mean solstitial colure, being $9''$, and the less semiaxis *b*P, $6''.7$. Let ABCD be the circumscribing circle, and let M*t*Q be perpendicular to *a*P; the point M in the circle corresponding to *t* in the ellipse. The pole *t* moves in a direction contrary to the order of the signs, following the moon's ascending node at the distance of a quadrant (reckoning from the corresponding point in the circle,) and consequently completing its revolution in about nineteen years. Thus, when the moon's node is in the

beginning of Aries, the pole is at a, and there is then the greatest obliquity of the ecliptic (being measured by Ea,) but no variation in longitude: When the node is in the beginning of Capricorn, the pole is at b, and there is then the greatest increment of longitude, but no variation in the obliquity of the ecliptic, &c. The longitude of M, then, is 90° more than the longitude of the node.

It is required to find the variation of the obliquity of the ecliptic, and the variation of longitude (or of the equinoctial points,) when the pole is at any point t of its orbit.

Let N=longitude of the node, then MPA=90°+N, and because of the great magnitude of Et in respect to tQ, PQ is the difference between Et and EP, or the variation of the obliquity; but R : cos N : : MP : PQ, therefore the variation PQ = 9″ cos N, by so much is the obliquity increased when the moon's node is in the ascending signs, or diminished when it is in the descending signs, cos N being negative in the latter case.

Again, the variation of the solstitial colure from its mean place is the angle PEt, which is measured on a parallel to the ecliptic by the small arc tQ; and since

$$9'' : 6''.7 : : \text{MQ} : t\text{Q} : : 9'' . \sin \text{N} : t\text{Q},$$

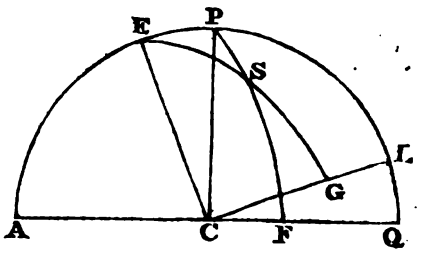
therefore tQ = 6″.7 . sin N, and the corresponding arc

on the ecliptic $= \frac{6''.7.\sin N}{\sin EP}$. So that the variation of the equinoctial points, and the consequent variation in the longitude of the stars is expressed by $-6''.7.\sin N$. cosec O, being a decrement when the node is in the northern signs, and an increment when it is in the southern.

PROBLEM XIX.

To find the variation in the right ascension and declination of a star, arising from the nutation of the earth's axis.

Let P be the pole of the equator ACQ, E the pole of the ecliptic CL, and S a star, of which the latitude is SG, and declination SF.



In the triangle EPS, the side ES being invariable, there are given the variations of the side EP and the angle E, to find those of the angle P and the side PS. Let A, D, N, O denote, as before, the star's right ascension and declination, the longitude of the moon's node, and the obliquity of the ecliptic: we have,

$$P'' = E''.(\cos EP - \sin EP.\cos P.\cot PS) - EP'' . \sin P . \cot PS,$$

and $PS'' = E''.\sin EP.\sin P + EP''.\cos P.$

From which, by substituting for E'' and EP'' their values as found in the preceding problem, (observing that PS'' and D'' must have contrary signs,) we have these formulæ,

$$A'' = -6''.7 \sin N.\cot O - 6''.7 \sin N.\sin A.\tan D - 9'' \cos N . \cos A . \tan D,$$

$$D'' = 6''.7 \sin N.\cos A + 9'' \cos N.\sin A,$$

which may also be resolved into the following :

$$A'' = -6''.7.\sin N.\cot O - 1''.15 \cos(A+N).\tan D - 7''.85 \cos(A-N).\tan D$$

$$D'' = 7''.85.\sin(A+N) + 1''.15.\sin(A-N).$$

PROBLEM XX.

To find the variation in the longitude and latitude of the stars, arising from the aberration of light.

If a star were placed in the pole of the ecliptic, (see Plate II. fig. 13.) it would appear, by the aberration of light, at the distance of 20″ from the pole: And the apparent place of the star, following the sun's place at the distance of a quadrant, would describe a

small circle of 40″ diameter, parallel to the ecliptic, about the true place, in the course of a year.

The apparent place of a star, in any other position, from the same cause describes, not a small circle of 40″ diameter, whose plane is parallel to the ecliptic, but (the orthographic projection of that circle on the surface of the sphere, viz.,) an ellipse; or, if the star be in the plane of the ecliptic, a straight line, or rather a very small arc.

Thus, let S be the true position of the star, and ES its distance from the pole of the ecliptic: And let the small circle described about S with the radius of 20″, on a plane parallel to the ecliptic, be projected into the ellipse *bcdf* on the surface of the sphere, (a very small portion of which may be considered as a plane). This curve *bcdf* is the apparent annual path of the star; its greater axis *bd*, parallel to the ecliptic and perpendicular to ES, is 40″; and its less axis *cf* (coincident with the circle of longitude ESD,) in proportion thereto, as sin lat. to rad.

Let the circumscribing circle BCDF represent the ecliptic. When the sun is at D, in conjunction with the star, or having the same longitude, the apparent place of the star is at *d*, and there is then the least apparent longitude (or greatest decrement thereof.) When the sun is at F, in the first quadrature after con-

junction, the apparent place of the star is at f, its greatest distance from the pole, &c.

Let R be the first point of Aries, a the apparent place of the star at any instant, and A the sun's place at the same time: then MaQ being perpendicular to bd, M is the point in the circle corresponding to a in the ellipse, and its distance from A is a quadrant. Let L=longitude of the star at S, and S=longitude of the sun at A, and the distance MC (being equal to AD,) will be expressed by (S ~ L). Hence $SQ = 20'' . \cos(S \sim L)$, and the arc of the ecliptic corresponding to SQ, or the angle $SEQ = 20'' . \cos(S \sim L) . \operatorname{cosec} ES$. So that if λ be the latitude of the star, its variation in longitude $= 20'' . \cos(S \sim L) . \sec \lambda$, being a decrement in the first and fourth quadrants of (S ~ L), and an increment in the second and third.

Again, because $Sb : Sf :: \text{rad} : \sin \lambda :: MQ : aQ$, the variation in latitude $aQ = 20'' . \sin \lambda . \sin(S \sim L)$.

Lastly, when the star is in the plane of the ecliptic, the apparent path is a straight line, there is no variation in latitude, and the variation in longitude $= 20'' . \cos(S \sim L)$.

PROBLEM XXI.

To find the variation in the right ascension and declination of the stars, arising from the aberration of light.

Let P (see Plate II. fig. 14.) be the pole of the equator ACQ, E the pole of the ecliptic CL, and S a star, of which the circle of right ascension PSF meets the ecliptic in M, and equator in F, and the circle of longitude ESG meets the ecliptic in G. Also, let K be the pole of PSF, and let the great circle through K, S meet the ecliptic in H.

In the triangle EPS, the side EP is supposed invariable, and the variations of the angle E and side ES are given (by the preceding problem,) to find those of the angle P and side PS.

1. For the variation in right ascension, we have,

$$P'' = ES''.\frac{\sin ESP}{\sin PS} - E''.\frac{\cos ESP.\sin ES}{\sin PS}$$ [450, Case 3.]

Let S denote, as before, the sun's longitude, L, λ, A, D the star's longitude, latitude, right ascension and declination, and O the obliquity of the ecliptic. It appears from the preceding problem, that when the sun is

PROBLEM XXII.

The orbit of a planet, and its mean anomaly at any instant being given, to find the true anomaly.

Let AMP (see Plate II. fig. 15.) be the orbit of the planet, AP the line of the apsides, C the centre of the ellipse, and S the focus in which the centre of the sun is supposed to be. Upon AP let the semicircle AEP be described: And in the time that the vector-radius SA, with the planet at A, moves round the focus S in the ecliptic orbit ABP, let the radius CA move uniformly round the centre C in the circle AEP. Let SM, CD be two cotemporary positions of these radii; the angle ASM is the true anomaly of the planet at M, and the angle ACD its mean anomaly.

Draw NMQ perpendicular to AP, join SD, SN and CN, and draw ST perpendicular to CN; the angle ACN is the eccentric anomaly.

Since both SA and CA describe areas proportional to the times, the sector AMS : semi-ellipse AMP : : sector ACD : semicircle AEP; hence, the sector AMS : sector ACD : : BC : CE : : MQ : QN : : sector AMS : sector ASN. Consequently, the sector ACD=ASN, and (taking ACN from both,) the sec-

tor DCN = triangle NCS, or $\frac{1}{2}$CN.DN = $\frac{1}{2}$CN.ST, whence the arc DN is equal to the perpendicular ST.

About D with the radius DC describe an arc to meet DS in *q*, and draw C*r* perpendicular to DS. The perpendicular ST, being equal to the arc DN, is greater than the sine of DN, and therefore the lines SD, CN meet when produced towards D, N. Hence the angle CDS exceeds the angle DCN, and therefore the arc C*q* is always greater than the arc DN.

Now, when the eccentric anomaly ACN is less than a right angle, the point S is farther from N than C is, therefore the distance of the lines SD, CN at S, that is, the perpendicular ST (or arc DN,) is greater than their distance at C, viz. C*r*. Thus the arc DN is in this case of an intermediate magnitude between the arc C*q* and its sine C*r*. And as these arc both given (because C*s*, CD, and the angle ACD are given,) and their difference is small, the value of DN will be known nearly. Also, when the eccentric anomaly ACN is greater than a right angle, though C*q* and C*r* are in this case both greater than DN, yet (on account of the smallness of the eccentricity of the planetary orbits,) C*r* will still be a near approximation to the value of DN.

In the present position of the straight line CN, the arc DN and the perpendiclar ST have no difference;

but let the line CN move about C into a position (*nt*) parallel to SD, then DN becomes equal to C*q*, and ST to C*r*, that is, the difference of DN, ST becomes equal to the difference of C*q*, C*r*. But the difference, thus generated, is composed of the small increment of DN, and the smaller decrement or increment of ST, which are to each other very nearly in the ratio of CN to CT. If, therefore, the given difference of C*q*, C*r* be divided in the ratio of CN : CT (: : R . CD : cos CSD . CS), and the greater part added to the given angle CSD, we shall have the eccentric anomaly ACN sufficiently correct.

In more eccentric orbits, after finding ACN in this manner, a greater degree of accuracy may be obtained by substituting in the above expression of the ratio CN : CT in place of CSD, a mean between CSD and ACN.

The true anomaly may be found from the eccentric by the following analogy:

$$AS : SP : : \tan^2 \tfrac{1}{2}ACN : \tan^2 \tfrac{1}{2}ASM.$$

PROBLEM XXIII.

To compute the phases of a lunar eclipse.

), E (see Plate I. fig. 3.) be the centres of the

sun and earth, and AC, BD, &c. rays from the circumference of the sun's disc, touching the earth at C, D, &c. and meeting one another beyond it (because the sun is bigger than the earth,) at some point H. The cone HCD bounded by these rays is the dark shadow of the earth, within which no part of the sun is visible.

Again, let the rays AIDL, BICK, &c. meeting one another at some point I between S, E, touch the earth at D, C, &c. there will be formed another cone KIL, indefinitely extended towards K, L, from which the two cones CID, CHD being subducted, the remaining space is the earth's penumbra, within which part only of the sun is visible.

When the moon therefore enters the penumbra, her brightness must be diminished, as she receives less light from the sun; and the nearer she approaches the dark shadow, the more obscure she becomes. But no part of the moon will be totally darkened or eclipsed, unless she enter the dark shadow of the earth.

Let FMG be a section of the conical shadow CHD, at the distance of the moon from the earth, perpendicular to the axis EH. The angle FEM is the apparent semidiameter of that section, as seen at the distance of the earth's centre; the angle CFE is the moon's horizontal parallax; the angle ACS is the sun's apparent semidiameter, and CSE his horizontal parallax.

Now the angle FEM=CFE—CHE, and CHE=ACS —CSE, therefore FEM=CFE+CSE—ACS; that is, the apparent semidiameter of the dark shadow at the moon (viewed at the distance of the centres of the earth and moon,) is equal to the sum of the horizontal parallaxes of the sun and moon lessened by the sun's semidiameter.

This apparent semidiameter of the shadow, if the place of observation be nearer than (C or) E, must be enlarged in the same proportion as the moon's semidiameter is augmented in such cases.

Now, if the moon's latitude at the time of opposition be less than the semidiameter of the shadow, there will be an eclipse of the moon, the phases of which may be thus computed.

Let FHG (see Plate I. fig. 4.) be the circular section of the earth's shadow at the moon, of which C is the centre, and the diameter FG is in the plane of the ecliptic. Let CM, perpendicular to FG, be the moon's latitude at the instant of opposition, CA the difference of longitude of the sun and moon for $1\frac{1}{2}^{h}$ before the opposition, and AB, perpendicular to FG, the moon's latitude at that time. Draw BMOP, which will represent the moon's path, make CD = CA, draw DE, EI parallel to AB, AD, and CN perpendicular to BP; make CP = sum, and CO = difference of the semi-

diameters of the shadow and the moon, and about the centres P, O, with a radius equal to the moon's semi-diameter, describe two circles touching the former at R, Q.

If the moon be moving from the node at L towards H, the eclipse begins when her centre is at P; at O is the beginning of the total obscuration, and at N is the middle of the eclipse.

In the right-angled triangle EBI, EI, IB being given, we may find the angle BEI and the hypotenuse BE, thus: EI : IB : : rad : tan BEI.

Rad : sec BEI : : EI : BE.

In the right-angled triangle CMN, we have CM, the moon's latitude at the opposition, and the angle MCN =BEI; hence to find CN, MN, and the time (t) that the moon takes to move through MN:

$$\left.\begin{array}{l}\text{Sec BEI : rad : : CM : CN.}\\ \text{Rad : tan BEI : : CN : MN.}\end{array}\right\} \text{and BE : MN : : } 3^h : t.$$

In the right-angled triangle PCN, we have PC, CN to find PN, and the time in which it is described.

$(PC+CN)(PC-CN)=PN^2$, and BE : PN : : 3^h : T.

In like manner ON, and the time in which it is described, may be found from the triangle OCN.

PROBLEM XXIV.

To compute the general phases of a solar eclipse.

Let S, M, E (see Plate I. fig. 5.) be the centres of the sun, moon, and earth, M being in the same straight line between S, E, or nearly so. Let the rays AC, BD, &c. from the circumference of the sun's disc, touch the moon at C, D, &c., and intersect at some point H beyond it. The cone HCD formed by these rays is the dark shadow of the moon; and the part FG of the earth's surface on which the moon's shadow falls is completely darkened. Again, let the rays AID, BIC, &c., intersecting at some point I between S, M, touch the moon at D, C, &c.; there will be formed another cone KIL, indefinitely extended towards K, L, from which the cones CID and CHD being taken away, the remaining space is the moon's penumbra, within which part only of the sun is visible.

Thus, to an observer on the earth within the moon's penumbra, there is a partial eclipse of the sun, which is the greater, the nearer he is to the dark shadow: To an observer within the dark shadow, there is a total ...se: And, when the dark cone does not reach the ..., to an observer at the place where its axis (pro-

duced,) meets the earth's surface, there is a central and annular eclipse; that is, an annulus or ring of the sun's disc appears to encompass the moon, the apparent diameter of the latter being in that case smaller than the apparent diameter of the former.

Now, to find the apparent semidiameters of the dark shadow and penumbra on the earth's disc, as seen from the distance of the moon's centre, we have

The angle HMF = CFM — CHM.

And KME=KCE=CEM+CIM=CEM+IHD+IBD.

But the angle CFM is the moon's apparent semidiameter, as seen from the earth's surface at F, the angle CHM or IHD is the sun's apparent semidiameter, as seen from the earth, and the angle IBD is the moon's apparent diameter, as seen from the sun, which is so very small a quantity that it may be neglected.

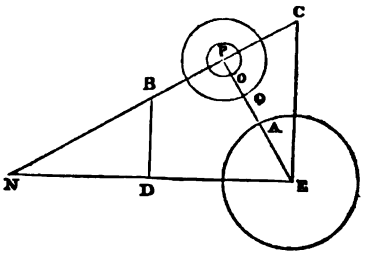
Hence, the apparent semidiameter of the moon's penumbra on the earth's disc (at the distance of the moon,) is the sum of the semidiameters of the sun and moon.

And the apparent semidiameter of the dark shadow on the earth (at the same distance,) is the difference of these semidiameters, when that of the moon is the greater.

Let S, E (see Plate I. fig. 6.) be the centres of the sun and earth, and L the centre of the moon at or

PROBLEM X.

To compute the general ... meet the ...bra, and let ...f the ecliptic. ...he disc, it is evi- ...ALE,) is the moon's ... ASE the sun's horizontal ...EL the true angular distance at E, the earth's centre. But the ...E — PSE, and PLE : PSE : : AME order, therefore, that the distance of the penumbra and disc may represent the ...ular distance of the sun and moon, in respect ...earth's centre, the semidiameter of the disc must ...express the difference of the sun's and moon's horizontal parallaxes, instead of the horizontal parallax of the moon simply: And other distances on the same plane must be reduced in the same proportion.



Hence, if EN represent the intersection of the ecliptic with the plane of the earth's disc, E, C the centres of the disc and penumbra in that plane at the time of conjunction, and CN the path of the penumbra's centre; then EP, perpendicular to CN, will be the nearest approach of these centres; and

happen only when EP is less than the
Q, the semidiameters of the disc and
P is less than EA, there is a cen-
total eclipse can take place only
semidiameter is greater than the sun's,
me time EP less than the sum of AE and
e semidiameters of the disc and dark shadow.

To find EP, CE being the moon's latitude at the time of the true conjunction, let BD be a less latitude at a given time before or after, and DE her longitude from the sun at that time,

$$\text{DE} : \text{CE} - \text{BD} : : \text{rad} : \tan x,$$
$$\text{and rad} : \cos x : : \text{CE} : \text{EP}.$$

Thus, if H, h be the moon's and sun's horizontal parallaxes, and M, S their semidiameters, the latter as seen from the moon, the former from the nearest surface of the earth, then EP, found as above, must be less than $(\text{H}+\text{S}+\text{M}-\frac{3}{2}h)$, for $(\text{S}+\text{M})$ being nearly $=\frac{1}{2}\text{H}$, must be diminished by $\frac{1}{2}h$, otherwise no eclipse can happen. Hence the general limit of solar eclipses, viz. the moon's latitude at the time of conjunction must be less than $1\frac{1}{2}^{\circ}$.

The phases of the moon's penumbra (and dark shadow when the moon's diameter is greater than the sun's,) moving over the earth's disc, may be computed in the very same manner as those of a lunar eclipse.

Let AED (see Plate I. fig. 7.) be that diameter of the earth's disc which is in the plane of the ecliptic, EC perpendicular to DA, and equal to the moon's latitude at conjunction, GCOF the path of the penumbra's centre over the disc, G being its position at the beginning, O at the middle, and F at the end of the eclipse: Also, gOf, the path of the centre of the dark shadow (when the moon's diameter is greater than the sun's,) g, f being its positions at its first and last contacts with the disc. The times of these phases are found in the same way as in a lunar eclipse.

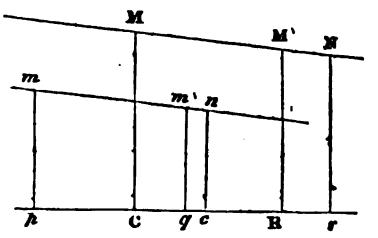
Let P be the pole of the equator, and produce EC, EP to meet the circumference of the disc in Q, R. Supposing EP to intersect the moon's path within the disc at H, it is plain, that when the penumbra's centre is at H, there is a central and total, or a central and annular eclipse at noon of the place H on the earth. To find the latitude and longitude of that place; in the triangle we have the hypotenuse PQ (the distance of the poles of the ecliptic and equator,) and the side PR (the moon's declination,) therefore QR or the angle CEH is given; but the angle CEO is known, therefore OEH will also be known. Hence we find OH, HE: from the former we have the time of the penumbra's centre being at H, which time, converted into degrees, gives the longitude of the place H from the meridian, for

which the tables are calculated; and from the latter, viz. HE, which is the sine of the sun's zenith distance, we obtain the latitude of the same place.

PROBLEM XXV.

To compute the phases of a solar eclipse for a given place.

On account of the moon's parallax, the conjunction of the sun and moon, as seen from a given place on the earth's surface, differs from the true conjunction which respects the centre of the earth. The former happens before or after the latter, according as the moon's longitude is east or west from the zenith of the given place. In the solution of this problem, therefore, the first thing to be done is to find the time of the visible conjunction.



1. Let C, M be the centres of the sun and moon at the true conjunction, CR being part of the ecliptic, and MM' part of the moon's orbit. Supposing the apparent conjunction to happen after the true, let M' be

the moon's true place 1^h after the true conjunction. Let m, m' be the apparent places of the moon corresponding to the former; Cp, Rq being the moon's parallaxes in longitude; mp, $m'q$ her apparent latitudes, and mm' her apparent orbit. Also, let c, n be the places of the sun and moon at the apparent conjunction.

Put $p =$ Cp, the moon's parallax at the time of the true conjunction, $q =$ Rq, her parallax 1^h after the true conjunction, $m =$ CR, the moon's hourly motion in longitude, $s =$ sun's hourly motion in longitude, $x =$ time between the true and apparent conjunction, and $y =$ Cc.

Then, since $pq : pc :: 1^h : x :: s : y$ nearly,

$$(p + m - q) : (p + y) :: 1 : x :: s : y.$$

Therefore, $(p + m - s - q) : s :: p : y$.

And $(p + m - s - q) : p :: 1 : x$.

Whence the value of x, or time of apparent conjunction, may be obtained nearly.

But, as we have supposed the moon's apparent motion, as well as her true motion, in longitude to be uniform, and neither of them is strictly so, but the former less uniform than the latter, so this value of x will not be accurate. It may be corrected thus: Let N be the moon's true place at the time of the apparent conjunction, and Cr her parallax in longitude at that time; in the time x the moon's true motion in longitude is Cr, and the sun's Cc.

Hence m : Cr : : 1 : x.

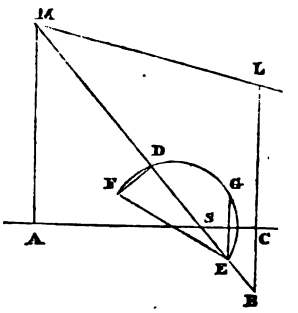
And s : Cc : : 1 : x.

Therefore $(m - s)$: cr : : 1 : x.

Now the moon's parallax in longitude, found for the time of apparent conjunction, obtained as above, will give cr very nearly: from the last analogy, therefore, x may be more accurately determined.

11. Next, to find the beginning of the eclipse.

Let C, L be the centres of the sun and moon at the time of the apparent conjunction, and S, M their positions at the beginning of the eclipse, MA being the moon's apparent latitude, and ML her apparent orbit.



Let l, d be the moon's apparent motions in longitude and latitude, s the sun's motion in longitude for the hour, or given time t, preceding the apparent conjunction, and c = sum of the two semidiameters. Draw MS = c, produce it to B, so that $l - s$: s : : MS : SB, bisect MB in D, and SB in E; make DF perpendicular to MB, and the angle DBF = inclination of the moon's apparent orbit to the ecliptic. Draw EF, upon which describe a semicircle, and apply EG = $(\frac{1}{2}a)$ $\frac{1}{2}$ the

moon at the apparent conjunction, suppose O, Q their places at the time required; let CN be perpendicular to NO, and OP to CQ. Putting CL$=a$, and $x=$ time between L, O, or C, Q; we have,

$$\lambda : l :: \mathrm{LO} : \mathrm{CP}.$$
$$l : s :: \mathrm{CP} : \mathrm{CQ}.$$
$$\text{And } \lambda : l :: \mathrm{CQ} : \mathrm{NO}.$$

Therefore $\lambda^2 : sl :: \mathrm{LO} : \mathrm{NO}$, and $\lambda^2 - sl : \lambda^2 :: \mathrm{LN} : \mathrm{LO}$;

but $\lambda : d :: \mathrm{CL} : \mathrm{LN}$,

and $\lambda : \mathrm{LO} :: t : x$.

Therefore $\lambda^2 - sl : ad :: t : x$,

$$\text{or, } \frac{(l-s).l}{d} + d : a :: t : x.$$

IV. To find OQ, and thence the magnitude of the eclipse, or the number of digits eclipsed, we have, as above,

$$(\lambda^2 - sl) : \lambda^2 :: \mathrm{LN} : \mathrm{LO};$$
$$\text{but } \lambda : l :: \mathrm{LO} : \mathrm{CP},$$
$$\text{and } l : (l-s) :: \mathrm{CP} : \mathrm{PQ}.$$

Therefore $(\lambda^2 - sl) : \lambda.(l-s) :: \mathrm{LN} : \mathrm{PQ} :: \mathrm{CL} : \mathrm{OQ}$.

Hence it appears that CN may generally be taken for the value of OQ, λ and l being very nearly equal. Then Qs, Om being the sun's and moon's semidiameters, we have $Qs + Om - \mathrm{OQ} = sm$ for the part of the

sun's diameter that is eclipsed; and Qs : *sm* : : 6° : digits required.

PROBLEM XXVI.

To project a solar eclipse for a given place.

Let S, L, C (see Plate II. fig. 8.) be the centres of the sun, moon, and earth at the time of conjunction, AVB the earth's disc, or circle of illumination, whose plane is perpendicular to SC, and intersects the ecliptic in the line ECF. Let S be the projecting point, and EVF the plane of projection, in which EF may be termed the horizontal line, and CVD (perpendicular to EF,) the vertical line: It is required to determine the projection of the moon's centre (or, which is the same, the position of P the penumbra's centre,) and the projection of a given place G (suppose Greenwich Observatory,) on the enlightened hemisphere, at a given instant, near the time of conjunction.

Let M be the position of the moon's centre at the given instant, M*m* her latitude, and *m*lN her orbit reduced to the ecliptic; then, supposing C (as well as S,) to remain at rest, we must suppose L to move with the difference of the velocities of L and C, that is, LM to

be the difference in longitude of the sun and moon at the given instant. Now C*n* is the projection of *lm* (or LM,) and P*n* the projection M*m*; and according as the moon's latitude is north or south, the point P is above or below ECF; also, according as her longitude is east or west of the conjunction, the point P is (to an observer at S,) on the right or left of CVD. Hence the position of P may be found thus:

As the difference of the sun's and moon's horizontal parallaxes,
Is to the moon's horizontal parallax;
So is their difference in longitude at the given instant,
To the distance of the penumbra's centre from the vertical line; and,
So is the moon's latitude at the given instant,
To the distance of the same from the horizontal line.

On account of the smallness of the arcs, we have here taken the latitude instead of sin latitude, and difference in longitude instead of sin difference in longitude × cos latitude to the radius of the moon's orbit.

Again, the position of the given place G on the enlightened hemisphere may be found for the given instant from the latitude and longitude of its zenith. Let GI, IC then be the arcs that express the latitude and longitude from the conjunction of the zenith to the ra-

dius CG: And the distance of the projection of G from EF will be determined by the sin GI, and its distance from CD by the sin IC × cos GI to the radius CG. Now, to the radius of the moon's orbit (used above,) the radius CG is equal to the moon's horizontal parallax for the given place, therefore,

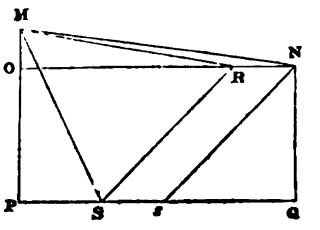
As radius is to the moon's horizontal parallax for the given place,
So is the latitude of the zenith to the distance of the projection of the given place from the horizontal line; and
So is the sine of the difference in longitude of the sun and zenith × cosine of the latitude of the zenith,
To its distance from the vertical line.

The projection of the given place is above or below EF, according as the latitude of the zenith is north or south; and on the right or left of CD, according as its longitude from the conjunction is east or west.

In this manner may the positions of G and P be found for any number of instants, and consequently the apparent path of each determined. Then, the instant in which the distance of two corresponding positions is equal to the sum of the sun's and moon's semidiameters is the beginning (or end) of the eclipse; and the instant in which such distance is the least possible, is the time of the greatest obscuration.

PROBLEM XXVII.

The beginning and end of a solar eclipse, or occultation of a fixed star, and the place of observation being given, to find the correction of the lunar tables of latitude and longitude.



Having computed, for the given instants, the moon's apparent latitudes, longitudes, and semidiameters, draw NO = difference of the apparent longitudes, MO (perpendicular to NO,) = difference of the apparent latitudes; make NR = sun's motion in longitude during the given interval of time, and join R, M. Upon RM describe the triangle RSM, of which the side MS shall be equal to the sum of the sun's and moon's semidiameters, when the moon has the greater apparent latitude, and the side RS equal to the sum of the semidiameters, at the time when she has the less apparent latitude. Through S draw PQ parallel to NO to meet the parallels MO, NQ in P, Q; then is MP the moon's greater apparent latitude that accords with the observation, and PS her apparent longitude from the sun at

the same instant. These being computed and compared with those before obtained from the tables, will shew the error (if any,) of latitude and longitude in these tables.

The calculation may be as follows: Put NO = D, MO = d, NR = s, MRO = x, and SMR = v: Also, MS = A, the sum of the semidiameters (the moon's being augmented according to her altitude,) at the time of the moon's having the greater apparent latitude, whether it be at the beginning or end of the eclipse, and SR = B, the sum of the semidiameters at the other given time. So that A and B differ from each other only by the difference of augmentation in the moon's semidiameters at the given times.

$(D-s):d::1:\tan x$, from the right-angled triangle ROM.

$(D-s):(A-B)::\cos x:\sin y$.
$(A+B):(D-s)::\sec x:\sin z$.
$2A:A+B::\cos y\cos z:\sin v$.
} from the triangle MSR, [471.]

$1:\sin(v+x)::A:MP$, the app. lat.
$1:\cos(v+x)::A:PS$, the app. diff. long.
} from the right-angled triangle MPS.

On account of the small difference between A and B, the angle y must be so very small that its cosine in the fourth analogy may be put equal to the radius. The second analogy, therefore, may be omitted.

In the case of an occultation of a star by the moon, the points R, N coincide, and NO, which is equal to PQ, a portion of the star's parallax in latitude, and not of the ecliptic, as in the preceding case, is found by reducing the apparent difference in longitude D in the proportion of radius to the cosine of the star's latitude, that is, NO = D . cos L : Consequently PS, when found, must be augmented in the same proportion, to get the moon's apparent difference in longitude from the star. Also, A and B will now express simply the moon's augmented semidiameters, observing that A is always taken to denote that semidiameter which the moon appears to have at her greater apparent latitude, whether it be at the instant of the star's immersion or of its emersion.

Hence, D.cos L : d : : 1 : tan x.

D.cos L : A+B : : cos x : sec z.

2A : A+B : : sin z : sin v.

1 : sin(v+x) : : A : greater app. diff. lat.

Cos L : cos(v+x) : : A : app. diff. long.

PROBLEM XXVIII.

Given three distances of a planet from the sun, and the angles between them, to find the eccentricity of its orbit.

In the orbit ADB, (see Plate II. fig. 9.) of which ASP is the line of the apsides, let there be given the three distances BS, CS, and DS, together with the angles BSC, and CSD, to find the ratio of the mean distance to the eccentricity.

Draw BD and CD, in which lines produced take BE : ED : : BS : SD, and CF : FD : : CS : SD; then the straight line which passes through the points E, F is the directrix of the ellipse, and (DN being perpendicular to it,) DN : DS is the ratio required.

To compute which, from the triangle BSD (because ES bisects the exterior vertical angle,) we have,

$$R : 2\sin \tfrac{1}{2}BSD : : \frac{BS \cdot SD}{BS - SD} : ES;$$ also from CSD,

$$R : 2\sin \tfrac{1}{2}CSD : : \frac{CS \cdot SD}{CS - SD} : FS.$$ [479.] And from

$$ESF \begin{cases} ES : FS : : R : \tan x. & [480.] \\ R : \tan(x+45) : : \tan \tfrac{1}{2}ESF : \tan(SEF + \tfrac{1}{2}ESF). \end{cases}$$

The line that bisects the angle BSD being at right angles to ES, and the line that bisects CSD at right angles to FS, the angle ESF is therefore equal to that contained by the bisecting lines, that is, $= \frac{1}{2}BSC$.

Again, $(BS+SD) : (BS-SD) : : \cot \frac{1}{2}BSD : \tan DES.$ [468.]

And $DN : DS : : (DN : DE + DE : DS) : : \sin DEN : R + \cos \frac{1}{2}BSD : \sin DES : : \sin DEN . \cos \frac{1}{2}BSD : R . \sin DES,$ where DEN, being equal to DES + SEF, is given.

Otherwise:

From the triangles BSD, CSD, we have,

$(BS + SD) : (BS - SD) :: \cot \frac{1}{2}BSD : \tan DES.$

$(CS + SD) : (CS - SD) :: \cot \frac{1}{2}CSD : \tan DFS.$

Then $EDF = DFS + ESF\ (\frac{1}{2}BSC) - DES.$

Again, from the triangle DEF, we have,

$(DE : DF ::) \cos \frac{1}{2}BSD \sin DFS : \cos \frac{1}{2}CSD . \sin DES :: R : \tan x.$

And $R : \tan (x + 45) :: \tan \frac{1}{2}EDF : \tan (DEF + \frac{1}{2}EDF).$

Whence, as before,

$DN : DS :: \cos \frac{1}{2}BSD \sin DEF : R \sin DES.$

THE END.

PLATE I

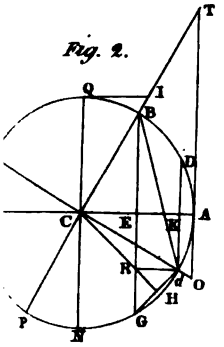


Fig. 2.

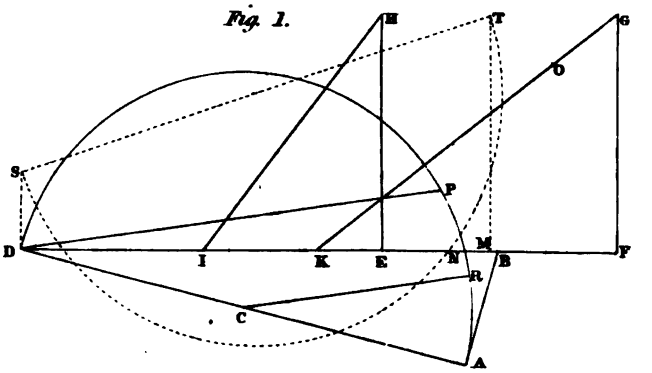


Fig. 1.

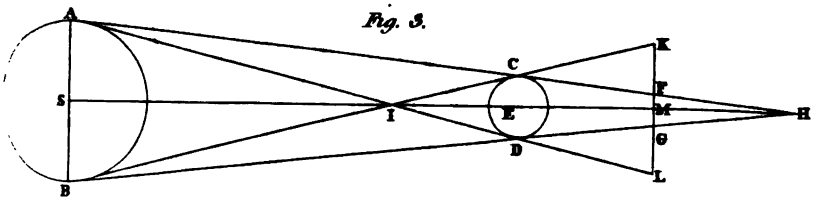


Fig. 3.

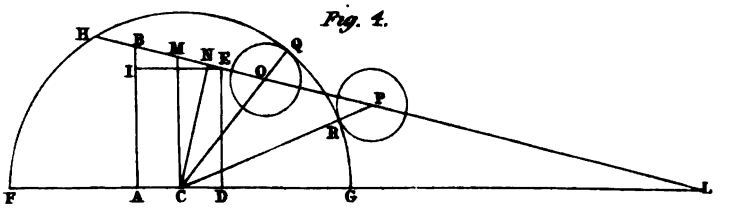


Fig. 4.

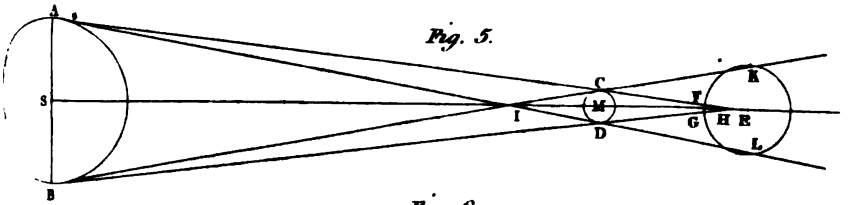


Fig. 5.

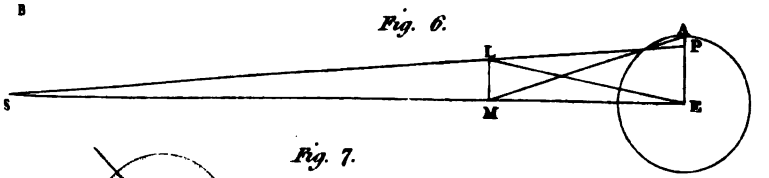


Fig. 6.

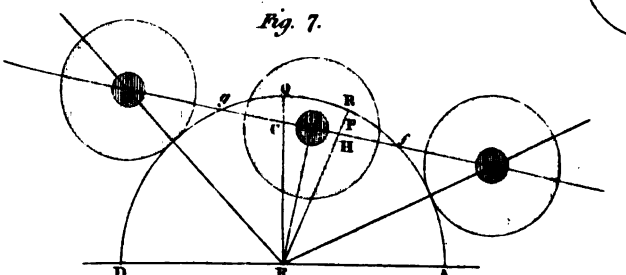


Fig. 7.

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PLATE II

Fig. 8.

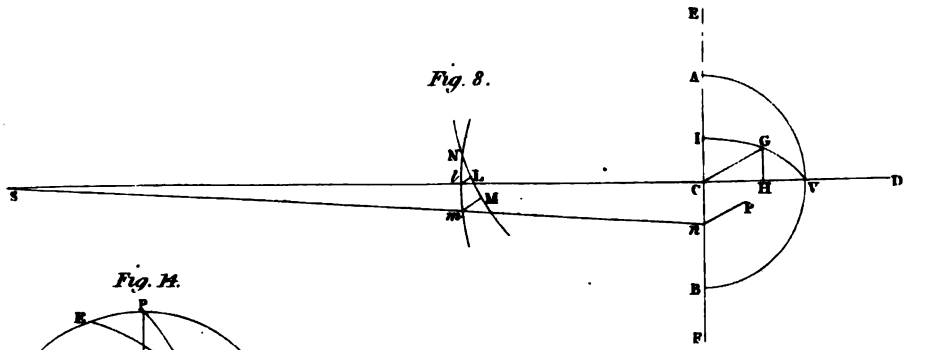


Fig. 14.

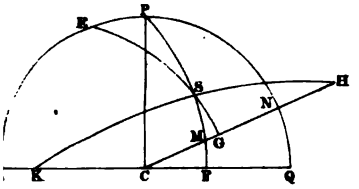


Fig. 10.

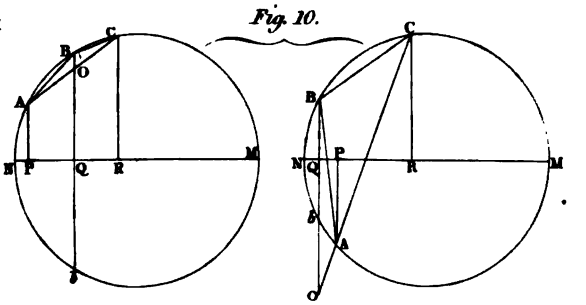


Fig. 9.

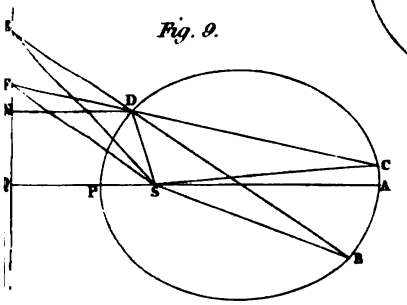


Fig. 11.

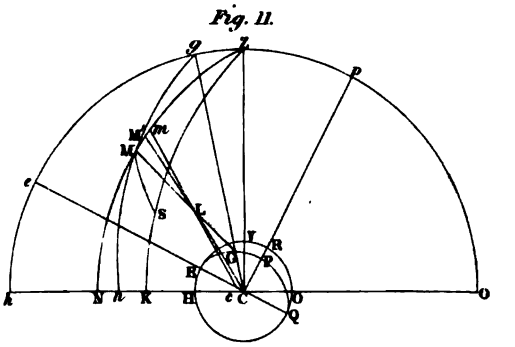


Fig. 12.

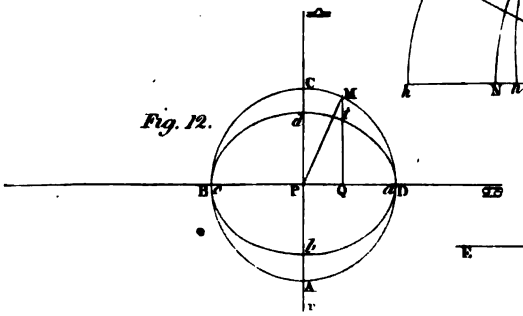


Fig. 13.

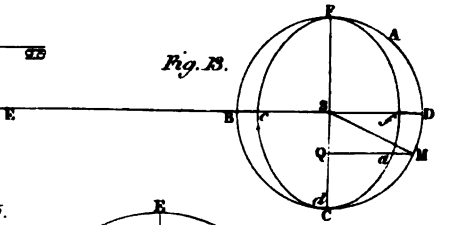
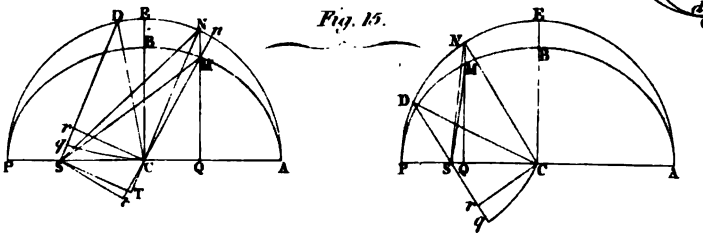


Fig. 15.



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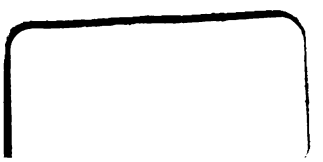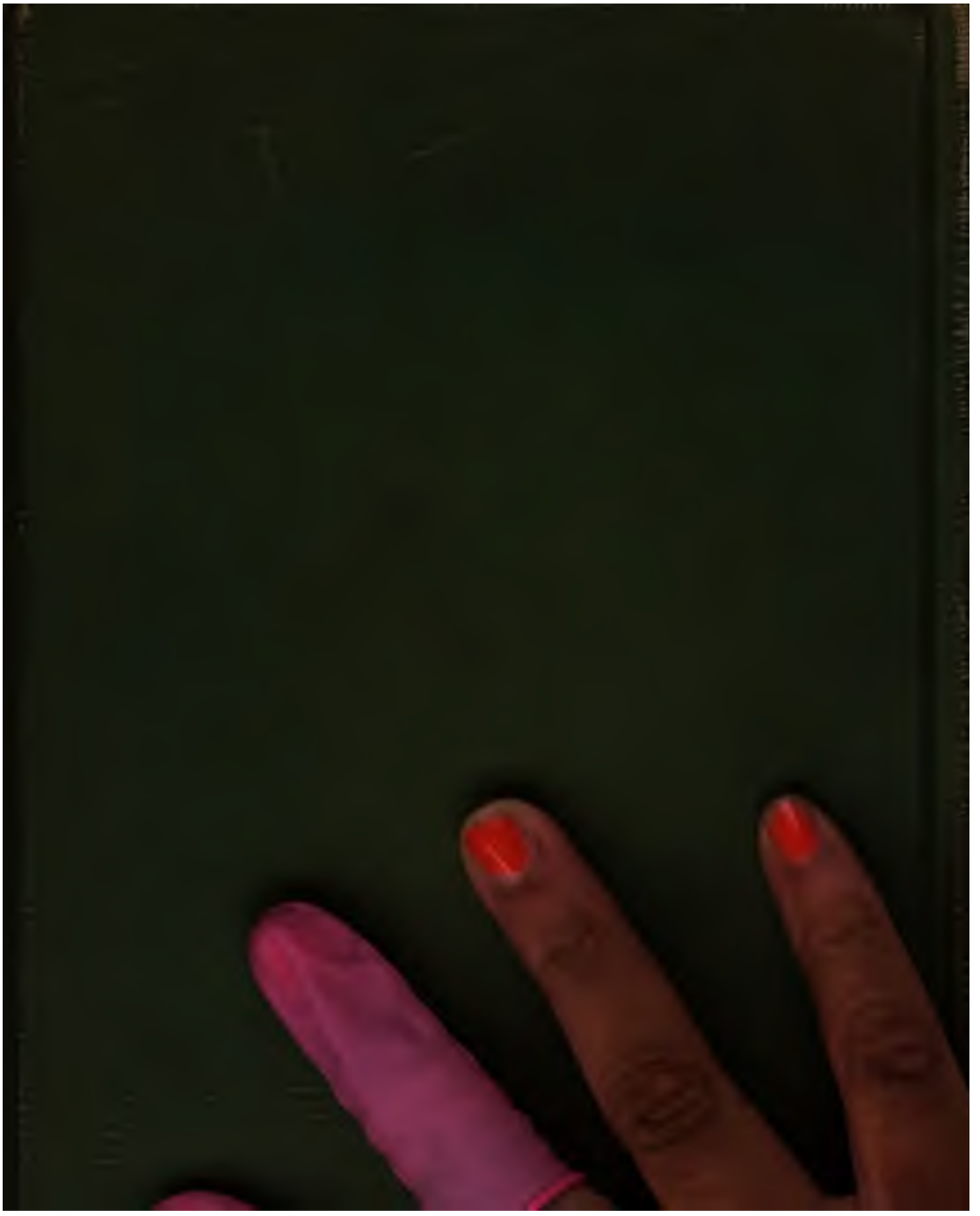
SEP 23 1937